Question 1 (Basic)

Evaluate:

$$\lim_{x \to 2}(3x^2 - 5x + 1)$$

Question 2 (Factorisation)

Evaluate:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

Question 3 (Challenging – Rationalisation)

Evaluate:

$$\lim_{x \to 0} \frac{\sqrt{x + 4} - 2}{x}$$

Question 1

$$\lim_{x \to 2}(3x^2 - 5x + 1)$$

Direct substitution:

$$3(2)^2 - 5(2) + 1 = 12 - 10 + 1 = 3$$

✅ **Answer: 3**

Question 2

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

Factorise:

$$\frac{(x-1)(x+1)}{x-1} = x + 1$$

Substitute:

$$1 + 1 = 2$$

✅ **Answer: 2**

Question 3

$$\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x}$$

Rationalise:

$$\frac{\sqrt{x+4} - 2}{x} \cdot \frac{\sqrt{x+4} + 2}{\sqrt{x+4} + 2} = \frac{x}{x(\sqrt{x+4} + 2)}$$

Cancel x:

$$\frac{1}{\sqrt{x+4} + 2}$$

Substitute:

$$\frac{1}{2+2} = \frac{1}{4}$$

✅ **Answer: $\frac{1}{4}$**

1. Average rate of change — 平均变化率
2. Instantaneous rate of change — 瞬时变化率
3. Derivative — 导数
4. Slope — 斜率
5. Secant line — 割线
6. Tangent line — 切线
7. Limit — 极限
8. Change — 变化量
9. Interval — 区间
10. Function — 函数

UNIT 2 KNOWLEDGE - CALCULUS 12 – DIFFERENTIATION: DEFINITION AND FUNDAMENTAL PROPERTIES

2.1 • Defining Average and Instantaneous Rates of Change at a Point

2.2 • Defining the Derivative of a Function and Using Derivative Notation

2.3 • Estimating Derivatives of a Function at a Point

2.4 Connecting Differentiability and Continuity: Determining When Derivatives Do and Do Not Exist

2.5 • Applying the Power Rule

2.6 • Derivative Rules: Constant, Sum, Difference, and Constant Multiple

2.7 • Derivatives of cos x, sin x, e^x , and ln x

2.8 • The Product Rule

2.9 • The Quotient Rule

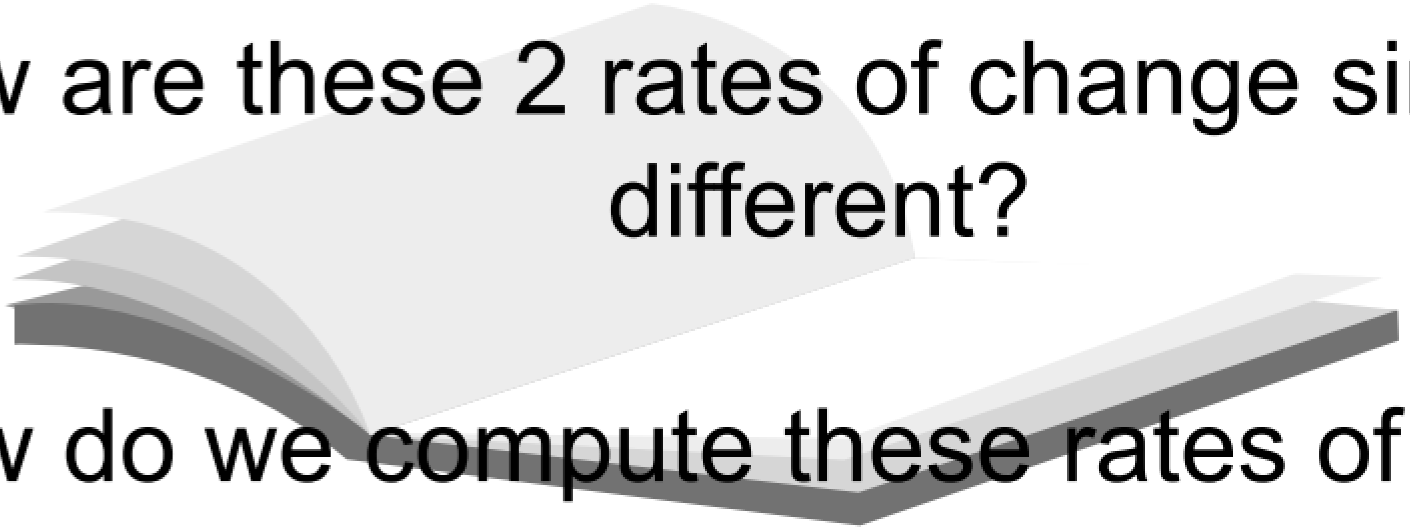
2.10 • Finding the Derivatives of Tangent, Cotangent, Secant, and/or Cosecant Functions

What Will We Learn? -> Unit 2 Topic 1

1. What is the average rate of change (AROC) of a function?

2. What is the instantaneous rate of change (IROC) of a function?

3. How are these 2 rates of change similar and different?



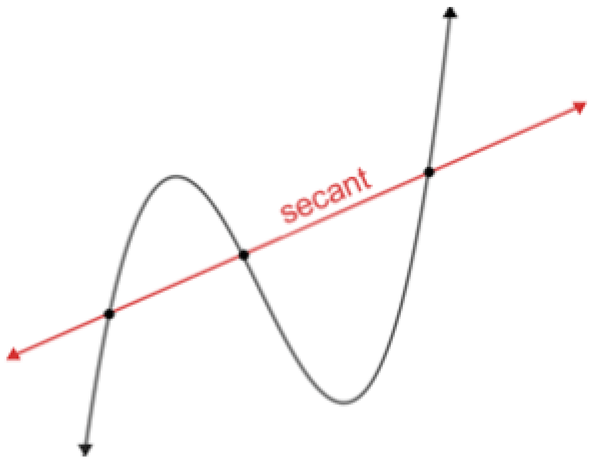
4. How do we compute these rates of change?

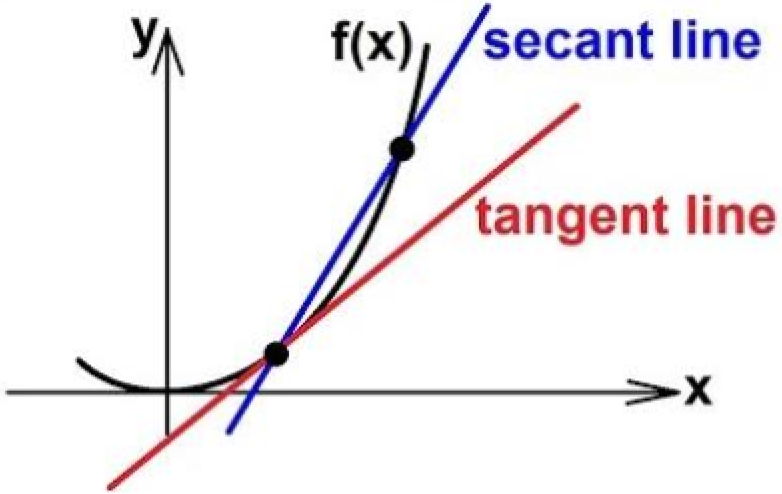
Key Terms

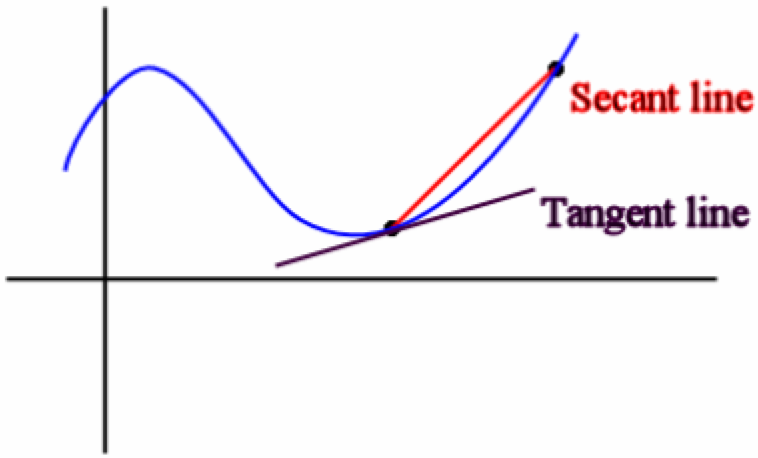
- **Secant line** - also simply called a secant, is a line passing through two points of a curve.

The word secant comes from the Latin word “secare”, meaning to cut.

- **Tangent line** - A line which locally touches a curve at **one and only one point**







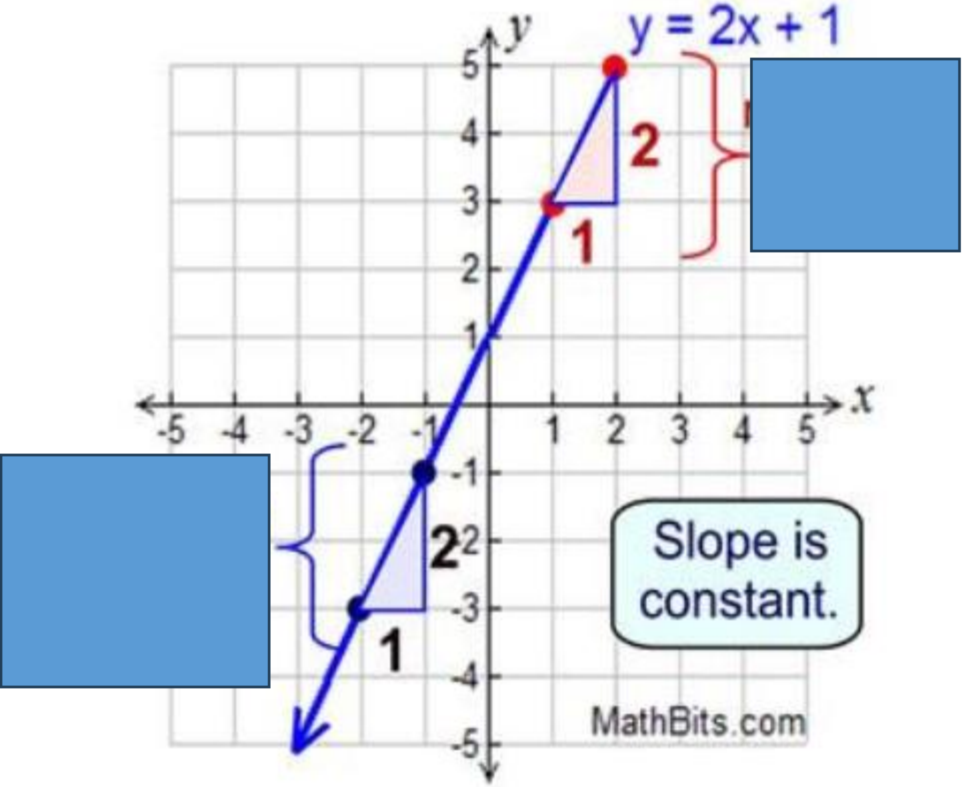
Let's start with "**rate of change**"

- This refers to a rate that describes how one quantity changes in relation to another quantity.

- For math functions in the $xy-plane$, it's specifically the rate at which $y(the\ dependent\ var$iable) changes for each unit of change in $x(the\ independent\ variable)$

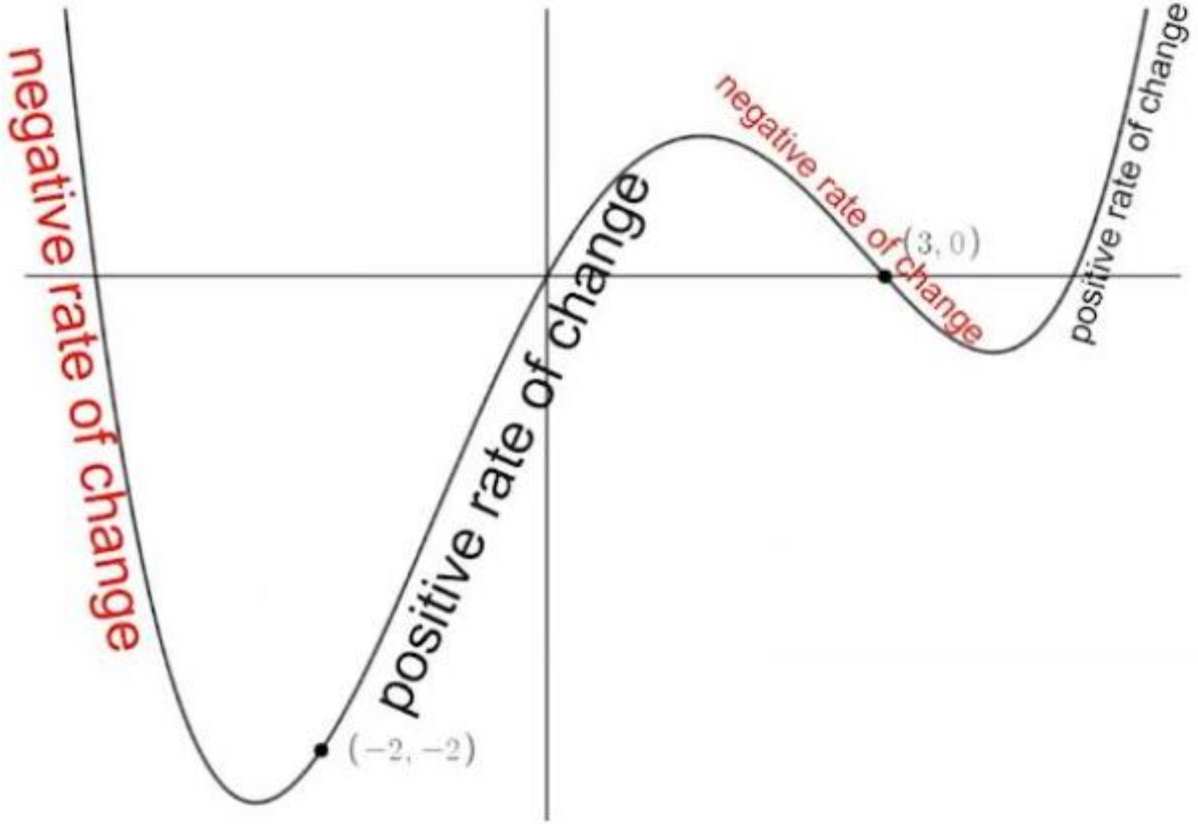
Rate of change

- For linear functions, the rate of change is constant.
- But what is the rate of change for non-linear functions?



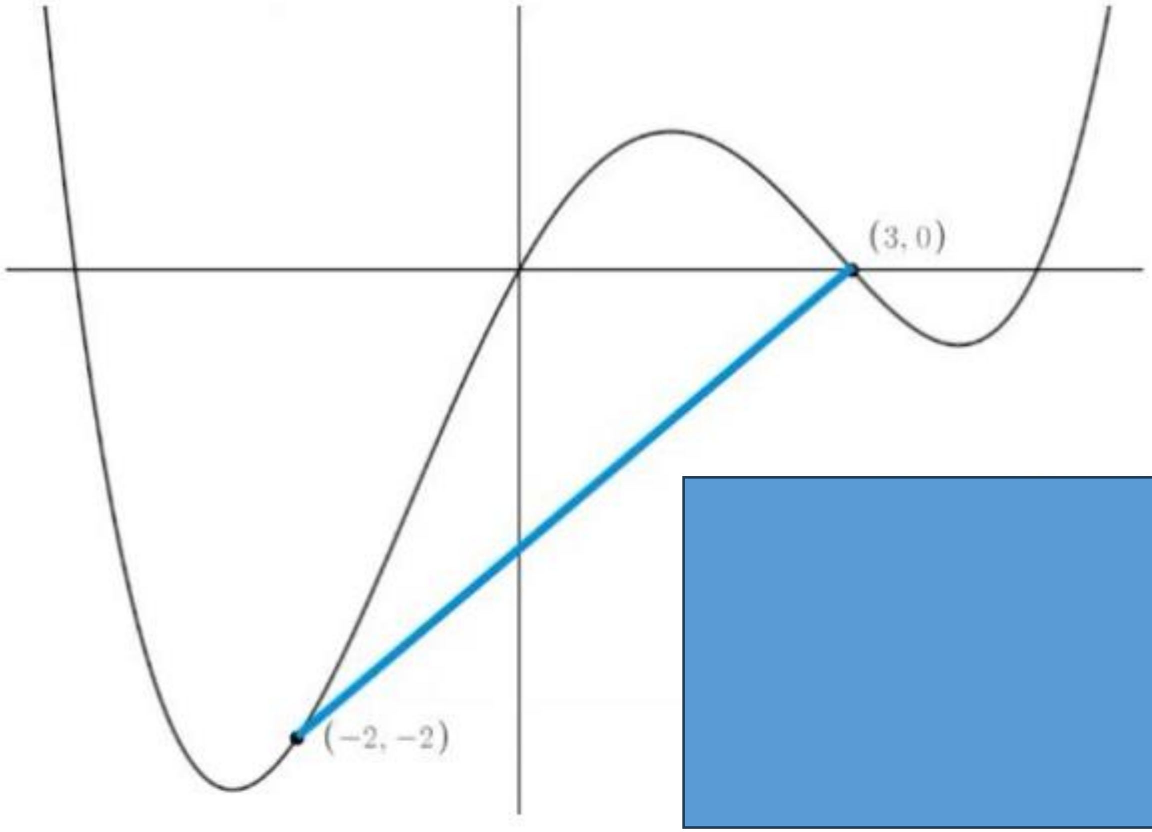
Rate of change for non-linear functions

- What is the rate of change for non-linear functions?
- It changes! So, we talk about what the average rate of change (AROC) equals ***ON AN INTERVAL.***
- E.g. from (-2,-2) to the point (3,0), what's happening to the function ***on the interval -2 to 3.***



Average Rate of Change (AROC)

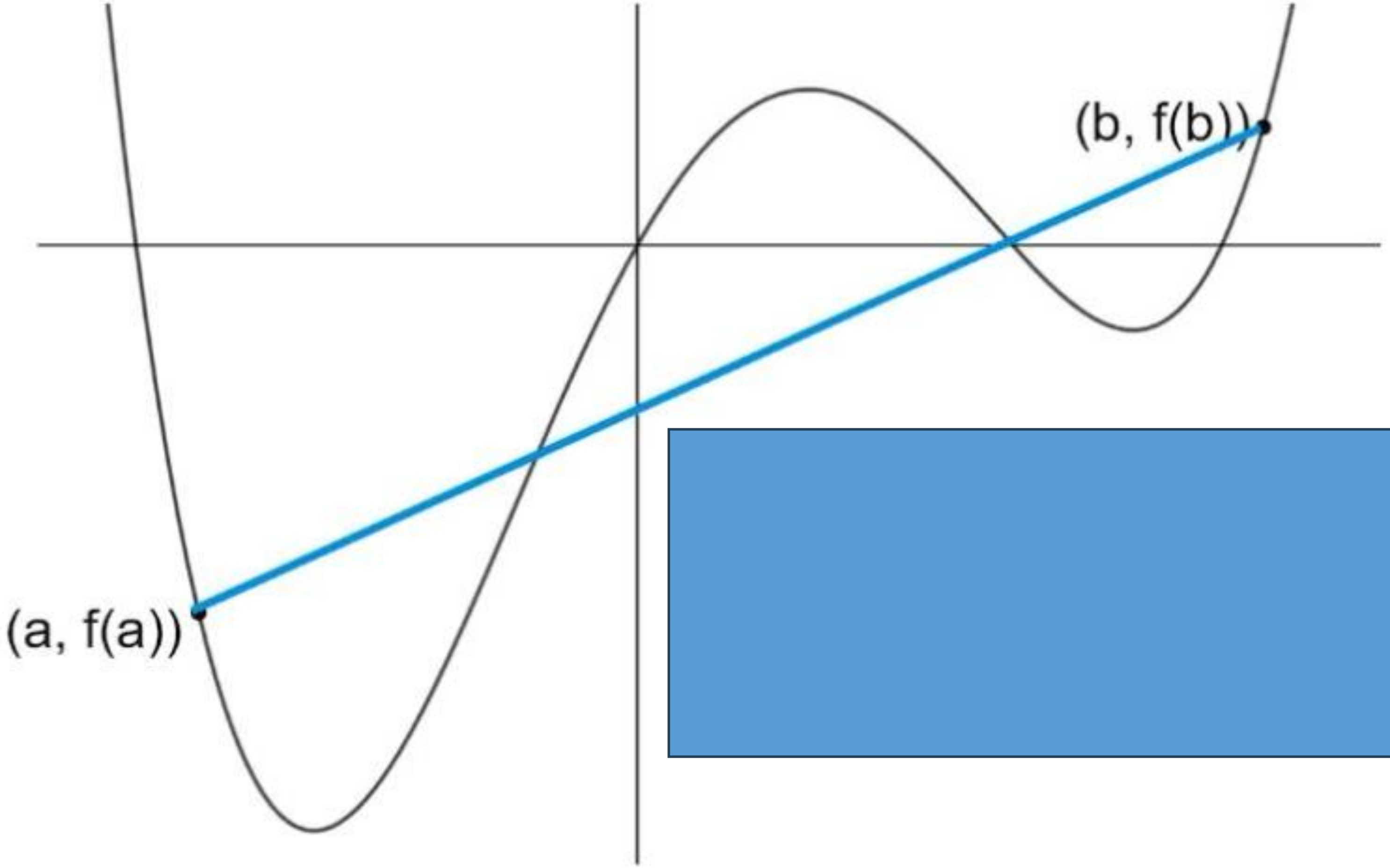
- How much does this function change on the closed interval [-2,3]?
- Let's look at the line segment
- **The AROC on this interval for this function will equal the slope of this secant line!**



Average Rate of Change (AROC)

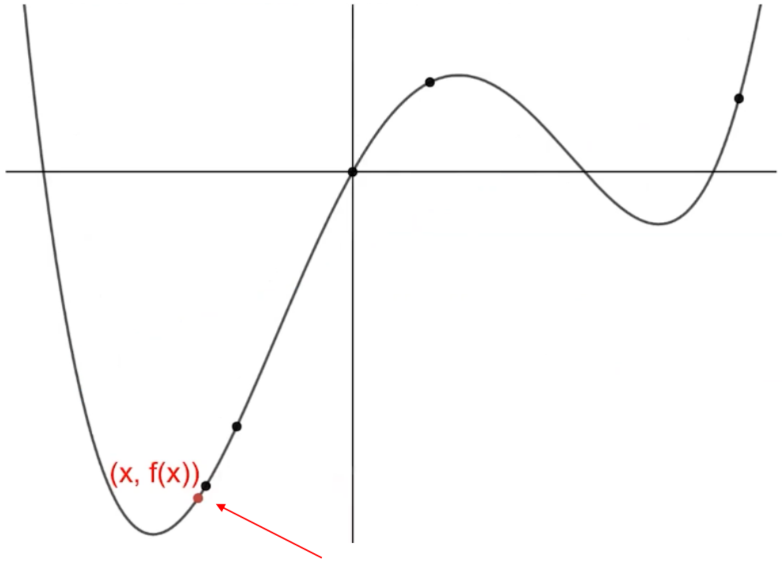
- So for any function, f(x) how can we express the **average** rate of change **on an interval**?

- Find the slope of the secant line connecting the 2 points.



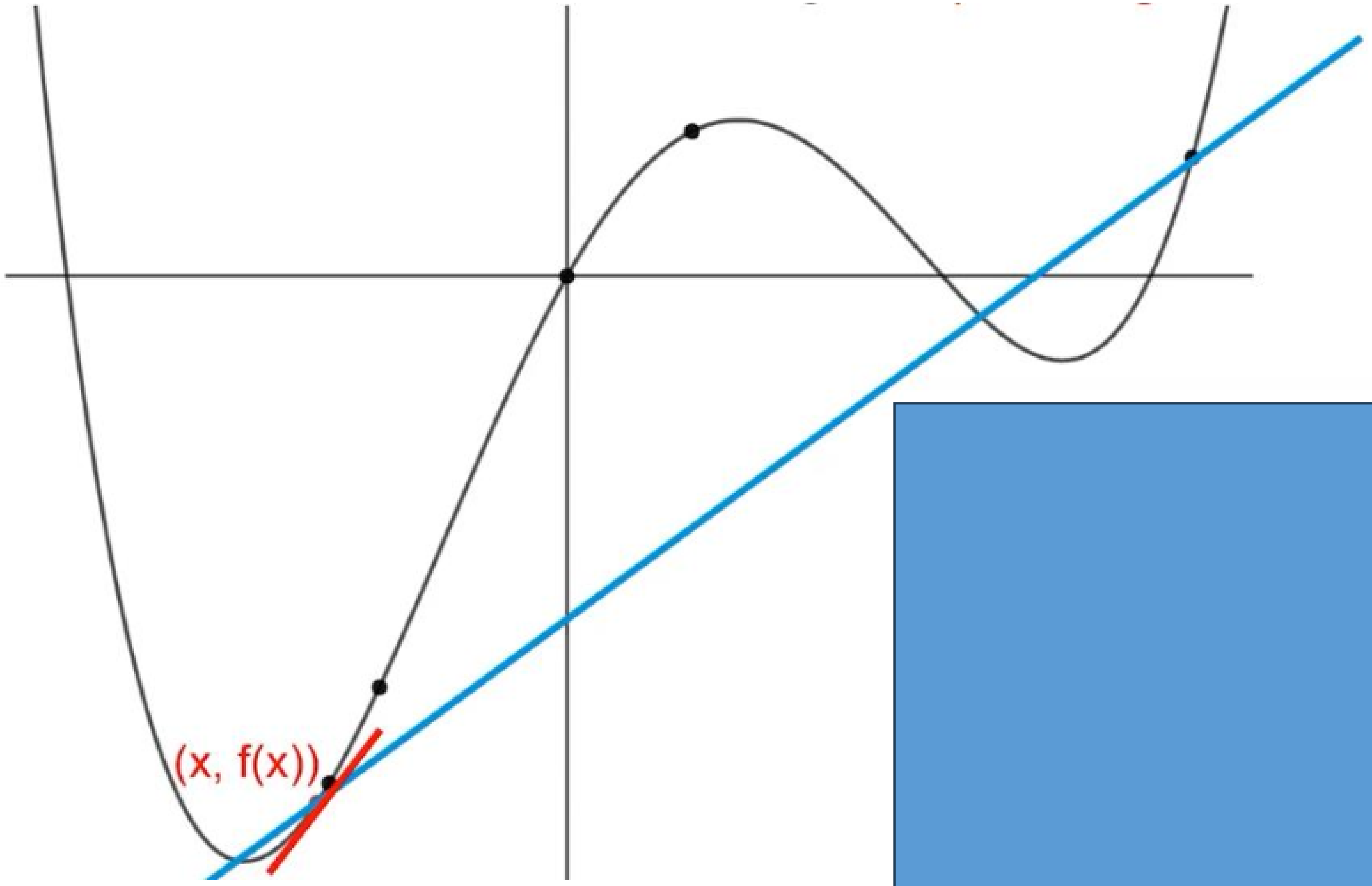
Instantaneous Rate of Change(IROC)

- For any function, f(x) how can we express the rate of change **AT A POINT**?
- We call this the **instantaneous** rate of change.



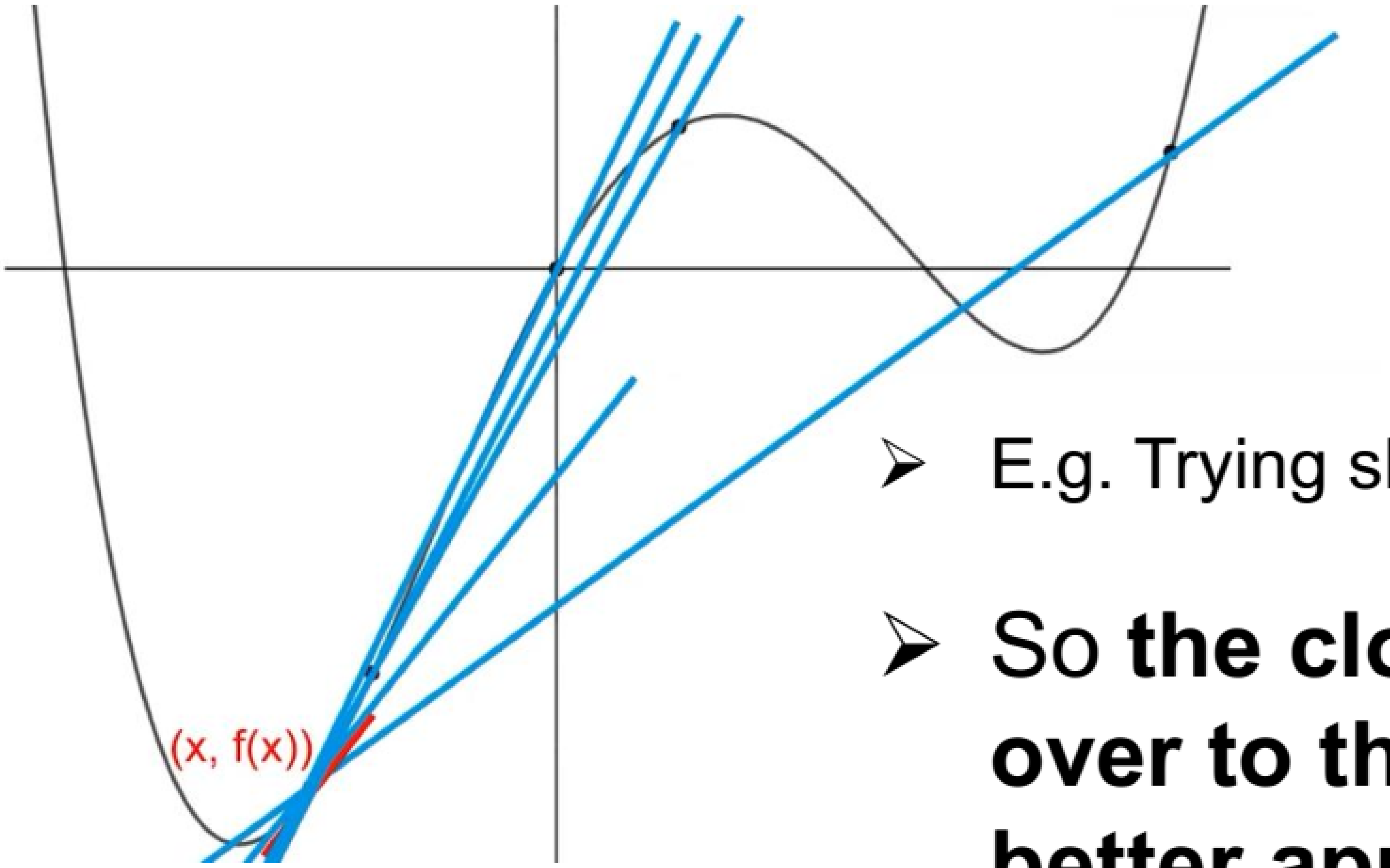
Instantaneous Rate of Change – AT A POINT

- We call this the **instantaneous** rate of change = slope of a tangent line.



Instantaneous Rate of Change – AT A POINT

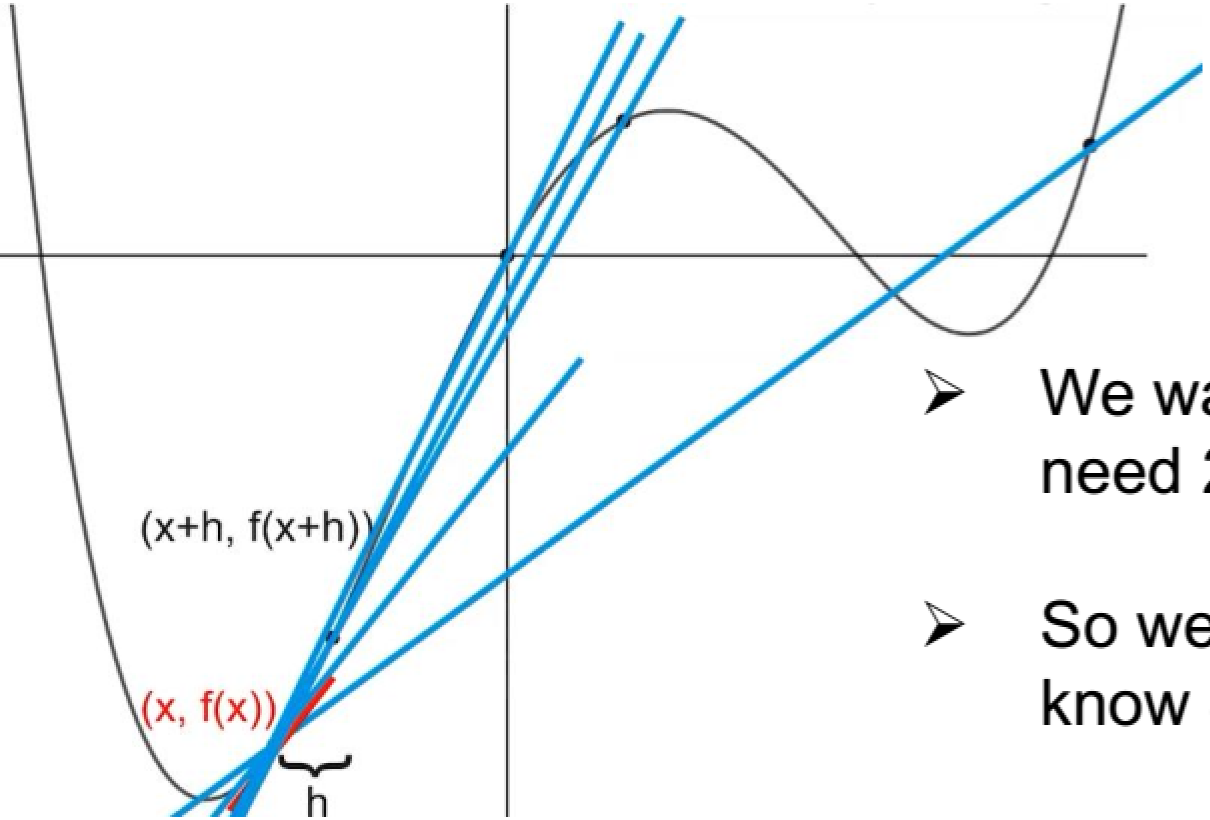
- We call this the **IROC** = slope of a tangent line.



- E.g. Trying slope of different secant lines.
- So **the closer we get that point over to the original point, the better approximation the slope of the blue (secant) line will be.**

Instantaneous Rate of Change (IROC) – AT A POINT

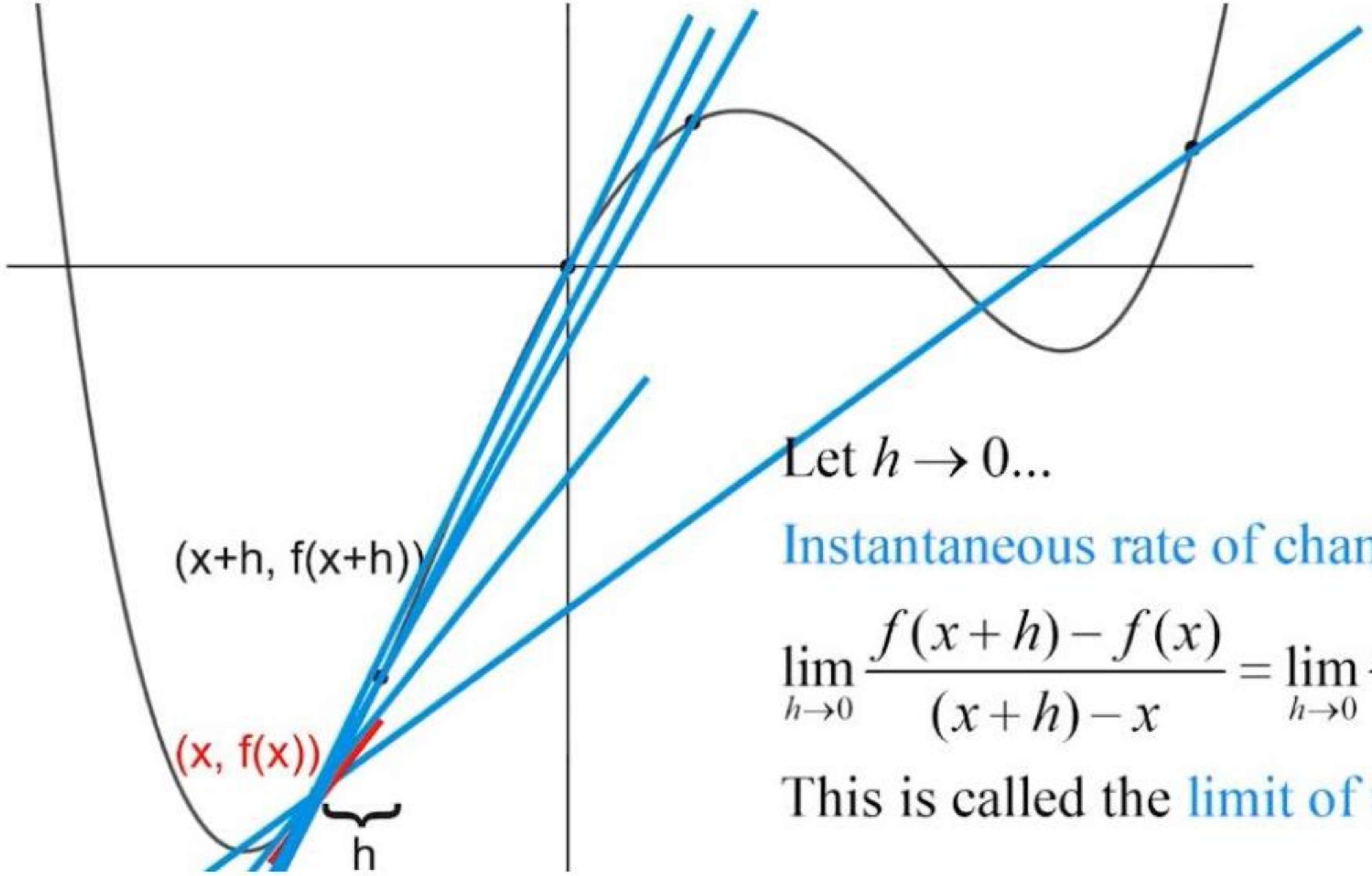
- We call this the **IROC** = slope of a tangent line.



- We want to know the slope of the red line, so we need 2 points somehow.
- So we can go a bit away from the point we want to know & call the distance **h.**
- We then let the distance **h** get smaller & smaller & bring the other points closer to x, f(x).
- We call it (x+h), f(x+h) –> 2nd point.

Instantaneous Rate of Change(IROC) – AT A POINT

- We call this the **IROC** = slope of a tangent line.



Let $h \to 0$...

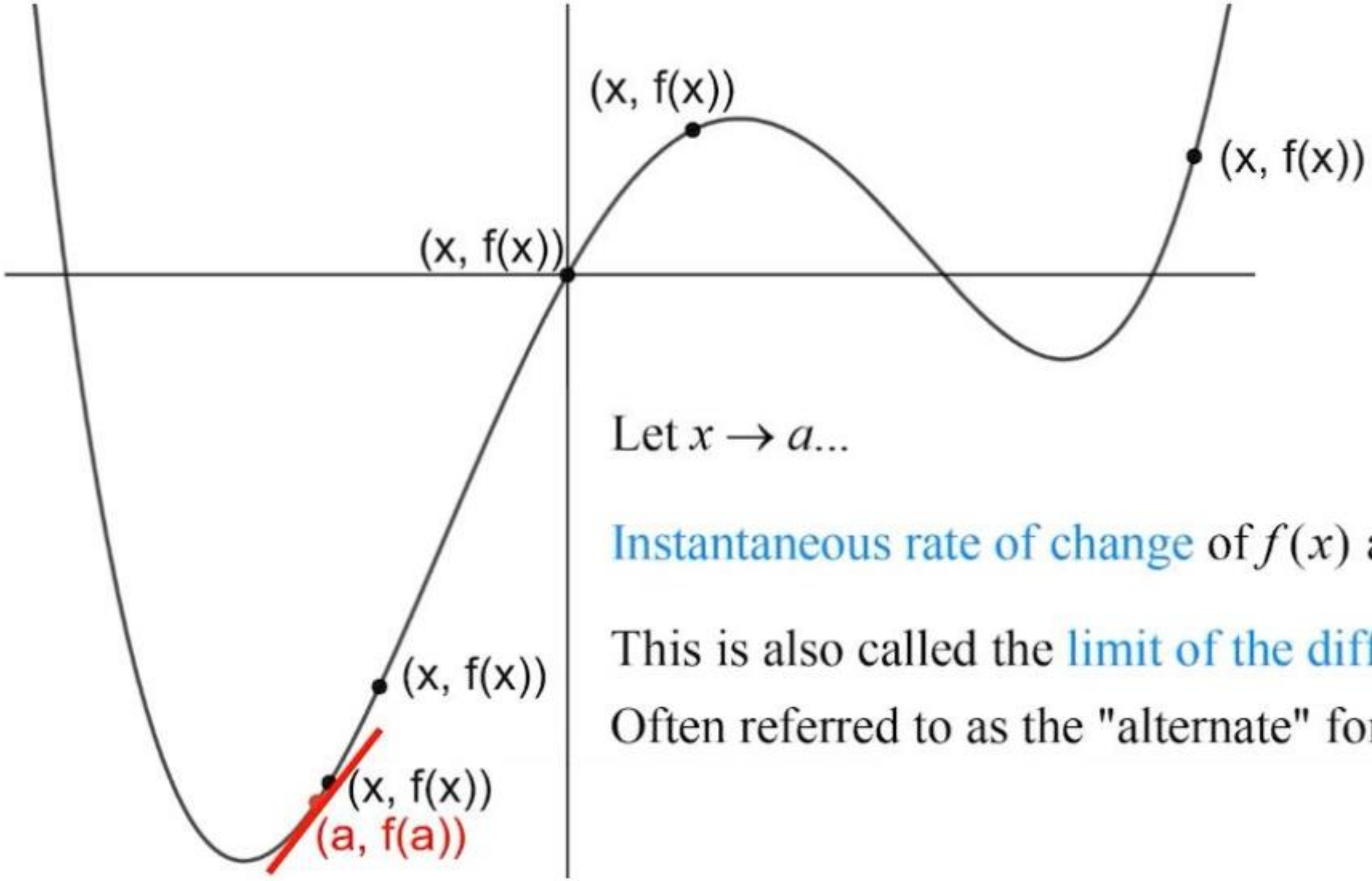
Instantaneous rate of change of $f(x)$ at $(x, f(x)) =$

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{(x+h)-x} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

This is called the limit of the difference quotient!

Instantaneous Rate of Change (IROC) – AT A POINT

- There is another form of this instantaneous rate of change….



Let $x \to a$...

Instantaneous rate of change of $f(x)$ at $(a, f(a)) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

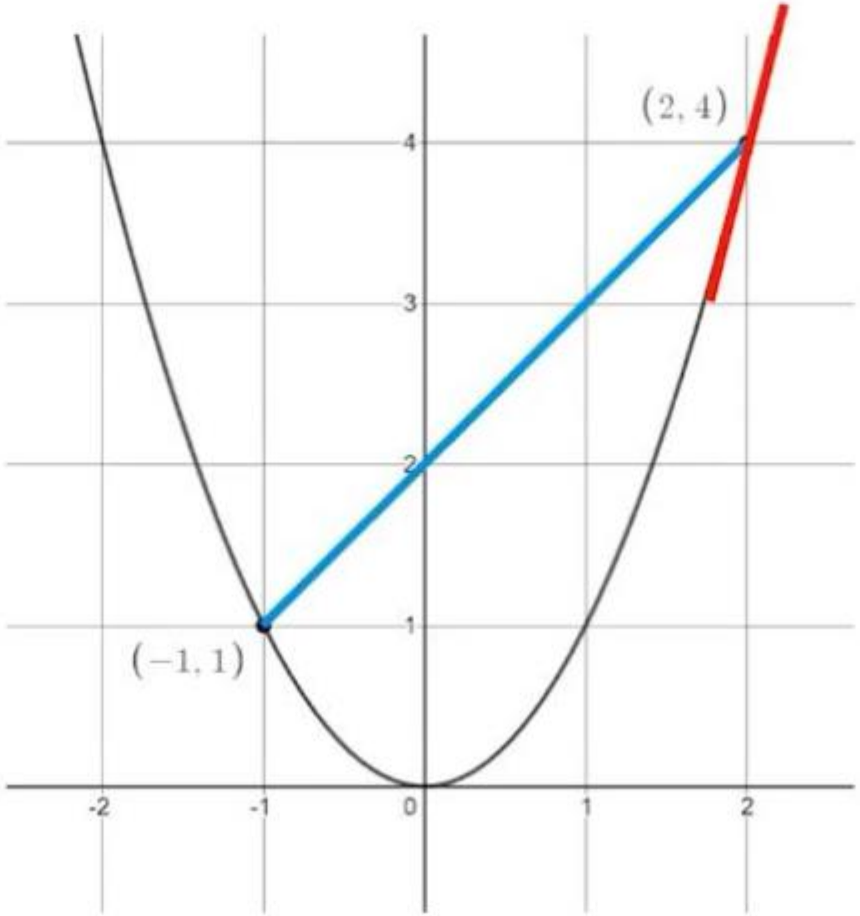
This is also called the limit of the difference quotient!

Often referred to as the "alternate" form of this limit definition.

Let’s see an example!

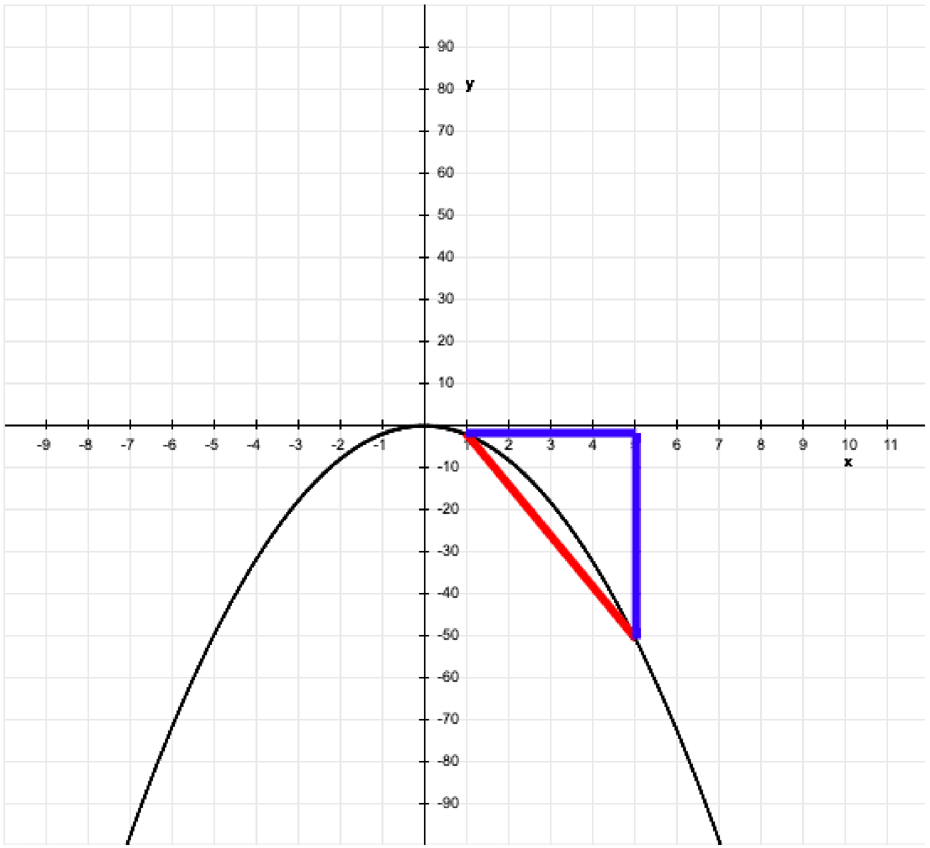
Finding AROC and IROC

The function, $f(x) = x^2$, is graphed below.
Find the average rate of change of $f(x)$ on [-1,2] and the instantaneous rate of change at the point (2,4).



- First Principles Example

- https://mathsforeveryone.freevar.com/IFY/calculus/firstprincipals.html



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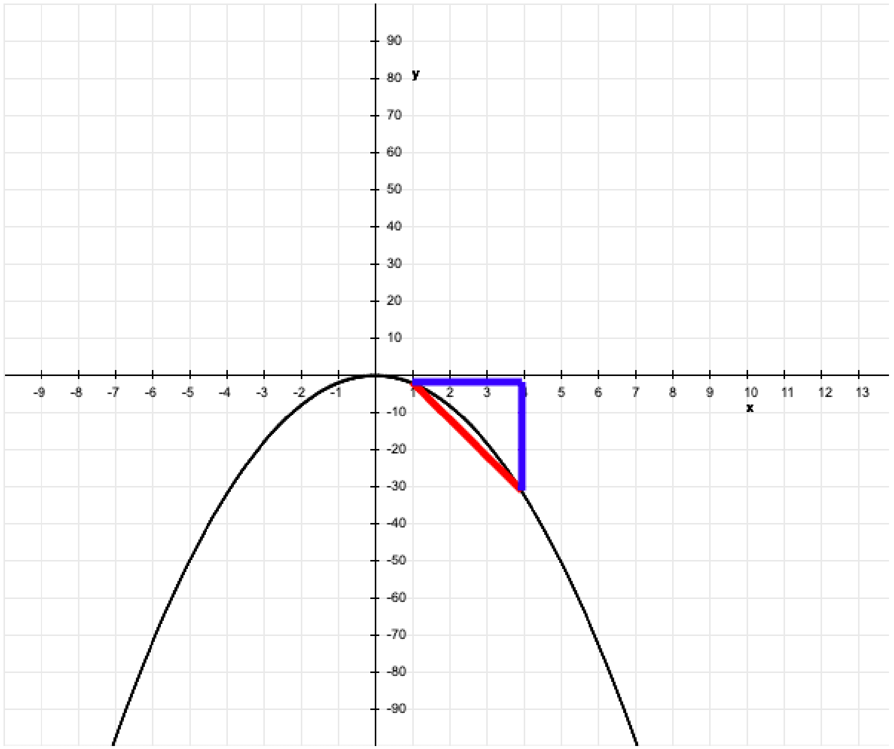

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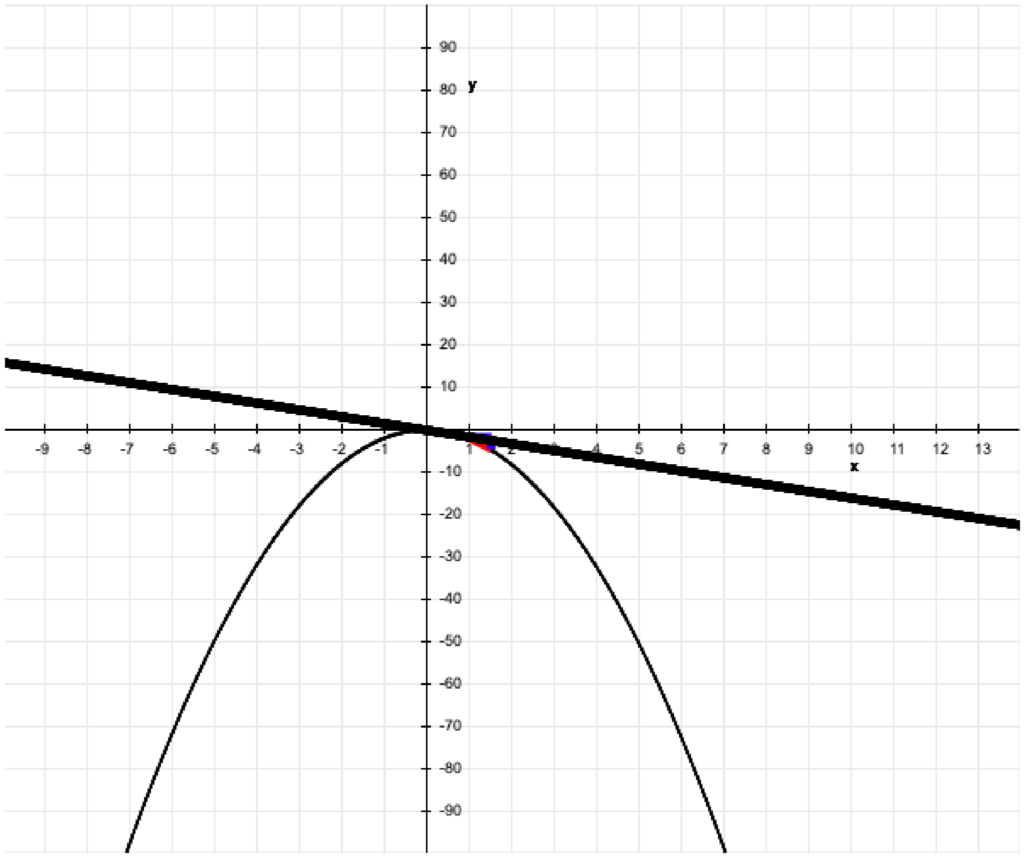
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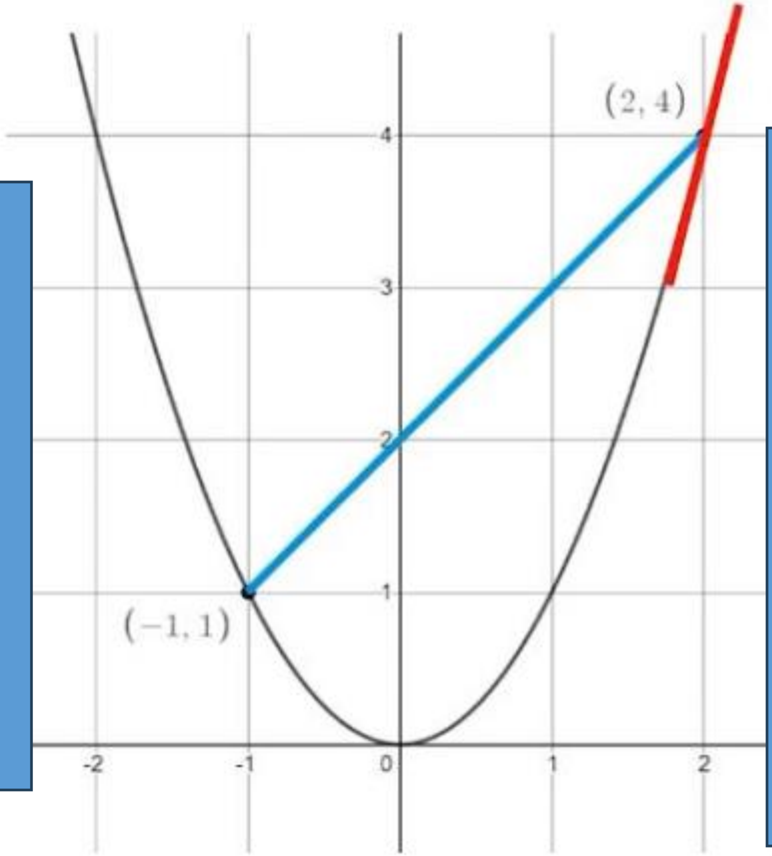
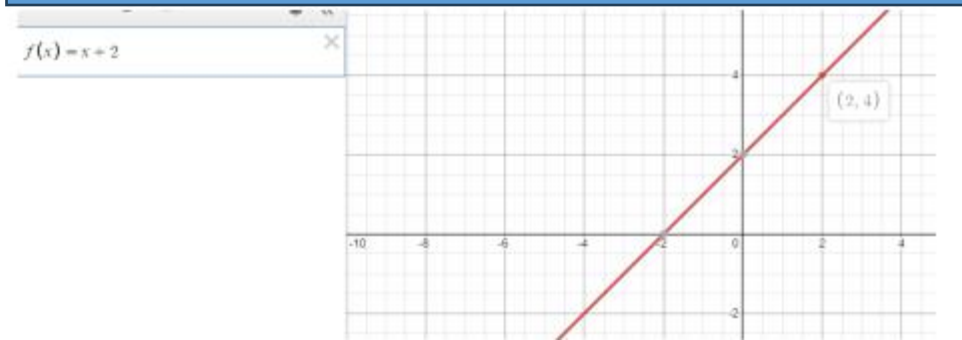
Finding AROC and IROC

The function, $f(x) = x^2$, is graphed below.
Find the average rate of change of $f(x)$ on [-1,2] and the instantaneous rate of change at the point (2,4).

Average rate of change of $f(x)$



Instantaneous rate of change of $f(x)$



Key Takeaways

- What is the average rate of change of a function?

  -slope of secant line on an interval

$$\frac{\Delta y}{\Delta x} = \frac{f(b)-f(a)}{b-a}$$

- What is the instantaneous rate of change of a function?

  -slope of tangent line at a point

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad \text{OR} \quad \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Next-> What Will We Learn?

- How can we calculate the average rate or change or the instantaneous rate of change for a function with given information tabular, graphical or analytical form?

Let’s see some examples!

Defining Average & Instantaneous Rates of Change at a Point

x	0	2	4	5
$f(x)$	2	-24	-84	-110

$f(x)$ is a continuous function having selected values of x in the table above. Find the average rate of change on the closed interval $[2,4]$ and use the data in the table to approximate the instantaneous rate of change of $f(x)$ at $x=1$.

<table>
<tr><th>AROC (i.e. slope of secant line on an interval)</th><th>IROC (i.e. slope of tangent line at a point)</th></tr>
<tr><td>

$$AROC = \frac{f(4)-f(2)}{4-2}$$

$$= \frac{-84-(-24)}{2}$$

$$= \frac{-60}{2}$$

$$= -30$$

</td><td>

$x = 1$ *is not in the table, so we approximate it with points closest to* 1 *i.e.* 0 & 2

$$IROC \approx \frac{f(2)-f(0)}{2-0}$$

$$\approx \frac{-24-2}{2}$$

$$\approx -13$$

</td></tr>
</table>

Defining Average & Instantaneous Rates of Change at a Point

Find the instantaneous rate of change of $f(x) = x - x^2$ at $x = -1$

Steps for method 1: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$	**Solution**
1. Apply the limit of the difference quotient with $x = -1$ $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$	
2. Rearrange and solve	
3. Cancel out h and $plug$ in 0	

Defining Average & Instantaneous Rates of Change at a Point

Find the instantaneous rate of change of $f(x) = x - x^2$ at $x = -1$

Steps for Method 2:	$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

Defining Average & Instantaneous Rates of Change at a Point

Find the instantaneous rate of change of $f(x) = x - x^2$ at $x = -1$

Method 1: $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$	**Method 2:** $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$
$N.B.\;\; a\ is\ the\ same\ as\ x, so\ a = -1$	

Defining Average & Instantaneous Rates of Change at a Point

x	1	3	5	7
$h(x)$	-20	0	10	16

Let $h(x)$ be a differentiable function with selected values given in the table above.

What is the average rate of change of $h(x)$ over the closed interval $1 \le x \le 7$?

A. $-\frac{2}{3}$

B. $\frac{1}{6}$

C. 5

D. 6

Defining Average & Instantaneous Rates of Change at a Point

If g is the function given by $g(x) = 2x^2 - x^4$,

then the average rate of change of g on the closed interval [1, 2] is

A. -9

B. -4

C. 9

D. 24

Defining Average & Instantaneous Rates of Change at a Point

Rainfall intensity is defined as the rate at which the height of the water layer covers the ground, measured in millimeters per minute. The rainfall intensity at a particular weather station is modeled by the function R defined by $R(t) = 3\cos\left(\frac{t}{2}\right)$ for $0 \le t \le 60$, where $R(t)$ is measured in millimeters per minute and t is measured in minutes. What is the average rate of change of the rainfall intensity over the time interval $5 \le t \le 40$? Indicate units of measure.

Defining Average & Instantaneous Rates of Change at a Point

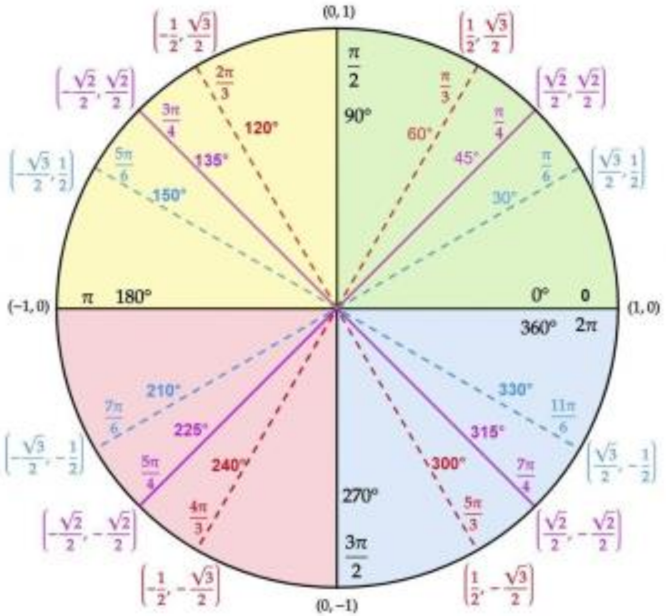
Let p be the function defined by $p(x) = e^{2x} \sin x$.

Find the average rate of change of p on the interval $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$.

$$AROC = \frac{P\left(\frac{3\pi}{2}\right) - P\left(\frac{\pi}{2}\right)}{\frac{3\pi}{2} - \frac{\pi}{2}} = \frac{e^{3\pi}\sin\left(\frac{3\pi}{2}\right) - e^{\pi}\sin\left(\frac{\pi}{2}\right)}{\pi}$$ as your answer

Or you can simplify it as $= \frac{-e^{3\pi} - e^{\pi}}{\pi}$

The Unit Circle Chart



Key Takeaways

- On free response questions, must show difference and quotient when finding the average rate of change.

- Free response answers do not need to be simplified.

TRUE or FALSE

The limit $h \to 0$ represents the slope of a tangent line.

TRUE

TRUE or FALSE

If the limit does not exist, the function is not differentiable at that point.

TRUE