TURN and TALK

Let

$$f(x) = \frac{1}{x-2}$$

(a) Find $\lim_{x \to 2^-} f(x)$

(b) Find $\lim_{x \to 2^+} f(x)$

(c) State whether the two-sided limit $\lim_{x \to 2} f(x)$ exists.

$f(x) = \frac{1}{x-2}$

(a) $\lim_{x \to 2^-} f(x) = -\infty$

(b) $\lim_{x \to 2^+} f(x) = +\infty$

(c) Two-sided limit does **not exist** (different infinities)

The Chain Rule

1. Chain rule — 链式法则
2. Composite function — 复合函数
3. Outer function — 外函数
4. Inner function — 内函数
5. Function of a function — 函数的函数
6. Differentiation — 求导
7. Derivative — 导数
8. Variable substitution — 变量代换
9. Nested function — 嵌套函数
10. Rate of change — 变化率

A reminder of the differentiation done so far!

- The gradient at a point on a curve is defined as the gradient of the tangent at that point.

- The function that gives the gradient of a curve at any point is called the gradient function.
- The process of finding the gradient function is called differentiating.

- The rules we have developed for differentiating are:

$$y = x^n \quad \Rightarrow \quad \frac{dy}{dx} = nx^{n-1}$$

Differentiating a function of a function

Let $y = (x^3 - 4)^2$

We can find $\frac{dy}{dx}$ by multiplying out the brackets:

$$y = x^6 - 8x^3 + 16 \Rightarrow \frac{dy}{dx} = 6x^5 - 24x^2$$

$$= 6x^2(x^3 - 4)$$

However, the chain rule will get us to the answer without needing to do this (essential if we had, for example, $(x^3 - 4)^{10}$.)

Consider again $y = (x^3 - 4)^2$

Let $u = x^3 - 4$ Then, $y = u^2$

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Differentiating both these expressions:

$$\frac{du}{dx} = 3x^2 \qquad \frac{dy}{du} = 2u = 2(x^3 - 4)$$

Now we can substitute for *u*

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Differentiating both these expressions:

$$\frac{du}{dx} = 3x^2 \qquad \frac{dy}{du} = 2u = 2(x^3 - 4)$$

Can you see how to get to the answer which we know is

$$\frac{dy}{dx} = 6x^2(x^3 - 4) \text{ ?}$$

We need to multiply $3x^2$ by $2(x^3 - 4)$

So, we have

$$\frac{du}{dx} = 3x^2 \qquad \frac{dy}{du} = 2u \quad = 2(x^3 - 4)$$

and to get $\frac{dy}{dx}$ we need to multiply $3x^2$ by $2(x^3 - 4)$

So,

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \times \frac{\cancel{du}}{dx}$$

This expression is behaving like fractions with the dus on the r,h,s, cancelling.

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Although these are <u>not</u> fractions, they come from taking the limit of the gradient, which <u>is</u> a fraction.

e.g. 1 Find $\frac{dy}{dx}$ if $y = (1 - 4x)^5$

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Solution:

Let $u = 1 - 4x$ Then $y = u^5$

Differentiating:

$$\frac{du}{dx} = -4 \qquad \frac{dy}{du} = 5u^4 = 5(1-4x)^4$$

Substitute for *u*

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \Rightarrow \frac{dy}{dx} = 5(1-4x)^4 \times -4$$

Tidy up by writing the constant first

$$= -20(1-4x)^4$$

We don't multiply out the brackets

e.g. 2 Find $\frac{dy}{dx}$ if $y = \frac{1}{\sqrt{2+5x}}$

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Solution: Whichever way we start we get

$$u = 2 + 5x \quad \text{and} \quad y = u^{-\frac{1}{2}}$$

$$\Rightarrow \quad \frac{du}{dx} = 5 \qquad \frac{dy}{du} = -\frac{1}{2}u^{-\frac{3}{2}}$$

$$= -\frac{1}{2}(2+5x)^{-\frac{3}{2}}$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}} \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{2}(2+5x)^{-\frac{3}{2}} \times 5$$

$$= -\frac{5}{2}(2+5x)^{-\frac{3}{2}}$$

SUMMARY

- The chain rule is used for differentiating functions of a function.

If $y = f(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Exercise

Use the chain rule to find $\frac{dy}{dx}$ for the following:

1. $y=(x^2-3)^3$
2. $y=(1-2x)^8$
3. $y=\sqrt{3x+4}$
4. $y=\dfrac{2}{1+3x}$
5. $y=\left(1+\dfrac{1}{x}\right)^5$

Solutions

2. $y = (1-2x)^8 \qquad u = 1-2x \quad \Rightarrow \quad y = u^8$

$$\frac{du}{dx} = -2 \qquad \frac{dy}{du} = 8u^7 = 8(1-2x)^7$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad \Rightarrow \quad \frac{dy}{dx} = -16(1-2x)^7$$

3. $y = \sqrt{3x+4} = (3x+4)^{\frac{1}{2}}$

$$u = 3x+4 \quad \Rightarrow \quad y = u^{\frac{1}{2}}$$

$$\frac{du}{dx} = 3 \qquad \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2}(3x+4)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{3}{2}(3x+4)^{-\frac{1}{2}}$$

Solutions

4. $$y = \frac{2}{1+3x} = 2(1+3x)^{-1}$$

$$u = 1+3x \quad \Rightarrow \quad y = 2u^{-1}$$

$$\frac{du}{dx} = 3 \qquad \frac{dy}{du} = -2u^{-2} = -2(1+3x)^{-2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = -6(1+3x)^{-2}$$

Solutions

5. $y = \left(1 + \dfrac{1}{x}\right)^5$

$$u = 1 + \frac{1}{x} \quad \Rightarrow \quad y = u^5$$

$$= 1 + x^{-1} \qquad \frac{dy}{du} = 5u^4 = 5\left(1 + \frac{1}{x}\right)^4$$

$$\frac{du}{dx} = -x^{-2} = -\frac{1}{x^2}$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{5}{x^2}\left(1 + \frac{1}{x}\right)^4$$

TIP: When you are practising the chain rule, try to write down the answer before writing out the rule in full. With some functions you will find you can do this easily.

However, be very careful. With some functions it's easy to make a mistake, so in an exam don't take chances. It's probably worth writing out the rule.

Chain Rule - Proof

Let $y = f(g(x))$.

Start from first principles:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

TRUE or FALSE

1. If $y = f(g(x))$, then $\frac{dy}{dx} = f'(x) \cdot g'(x)$.
2. The chain rule is used when one function is inside another function.
3. $\frac{d}{dx}[\sin(3x)] = \cos(3x)$.

Using the Chain and Power Rules

Find the derivative of $h(x) = \frac{1}{(3x^2+1)^2}$.

First, rewrite $h(x) = \frac{1}{(3x^2+1)^2} = (3x^2+1)^{-2}$.

Applying the power rule with $g(x) = 3x^2 + 1$, we have

$$h'(x) = -2(3x^2+1)^{-3}(6x).$$

Rewriting back to the original form gives us

$$h'(x) = \frac{-12x}{(3x^2+1)^3}.$$

Using the Chain and Power Rules with a Trigonometric Function

Find the derivative of $h(x) = \sin^3 x$.

First recall that $\sin^3 x = (\sin x)^3$, so we can rewrite $h(x) = \sin^3 x$ as $h(x) = (\sin x)^3$.

Applying the power rule with $g(x) = \sin x$, we obtain

$$h'(x) = 3(\sin x)^2 \cos x = 3\sin^2 x \cos x.$$

Using the Chain Rule on a General Cosine Function

Find the derivative of $h(x) = \cos(g(x))$.

Solution

Think of $h(x) = \cos(g(x))$ as $f(g(x))$ where $f(x) = \cos x$. Since $f'(x) = -\sin x$. we have $f'(g(x)) = -\sin(g(x))$. Then we do the following calculation.

$$\begin{aligned} h'(x) &= f'(g(x))\,g'(x) && \text{Apply the chain rule.} \\ &= -\sin(g(x))\,g'(x) && \text{Substitute } f'(g(x)) = -\sin(g(x)). \end{aligned}$$

Thus, the derivative of $h(x) = \cos(g(x))$ is given by $h'(x) = -\sin(g(x))\,g'(x)$.