Maximum and Minimum of Quadratics

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Turn and Talk

solve

$$x^4 - 5x^2 + 4 = 0$$

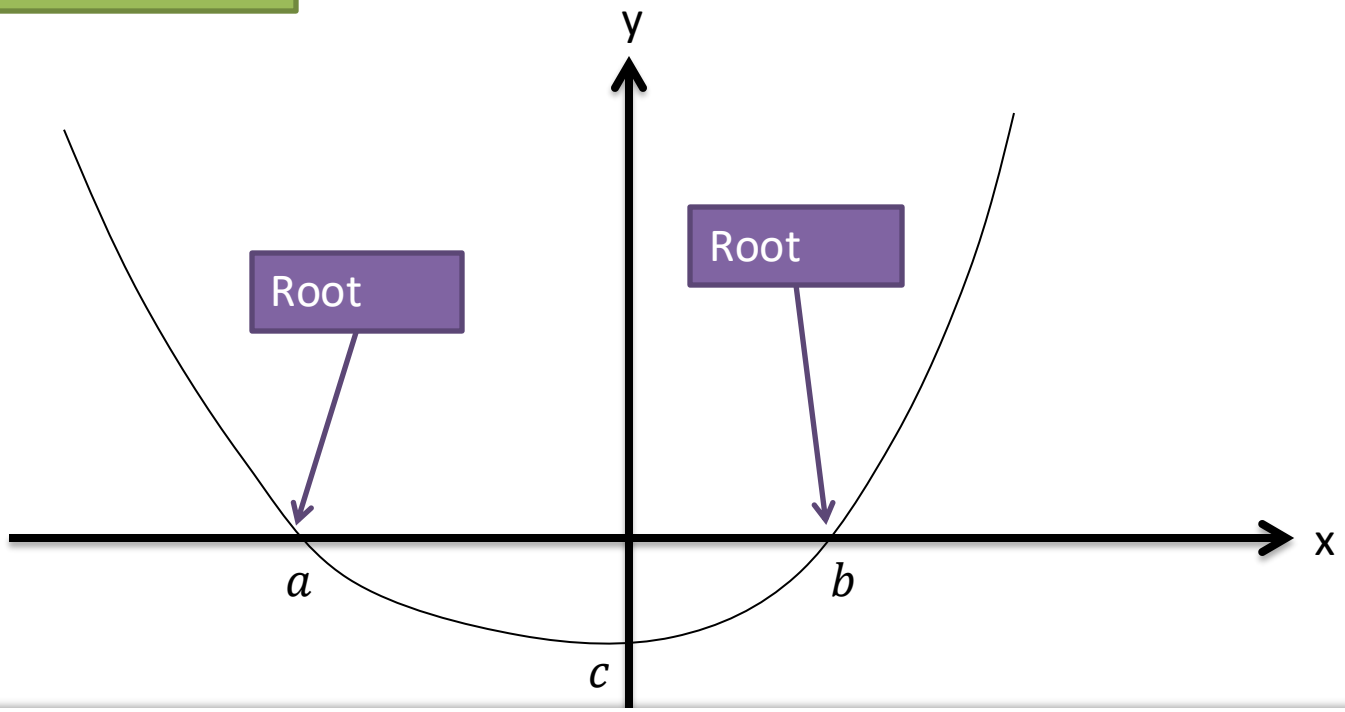
$x = -2, -1, 1, 2$

Maximum and Minimum

1. **Quadratic function** — 二次函数
2. **Vertex** — 顶点
3. **Maximum** — 最大值
4. **Minimum** — 最小值
5. **Axis of symmetry** — 对称轴
6. **Parabola** — 抛物线
7. **Opening upwards / downwards** — 向上/向下开口
8. **Coefficient a (leading term)** — 首项系数
9. **y-coordinate of vertex** — 顶点的纵坐标
10. **x-coordinate of vertex** — 顶点的横坐标

Key Terms

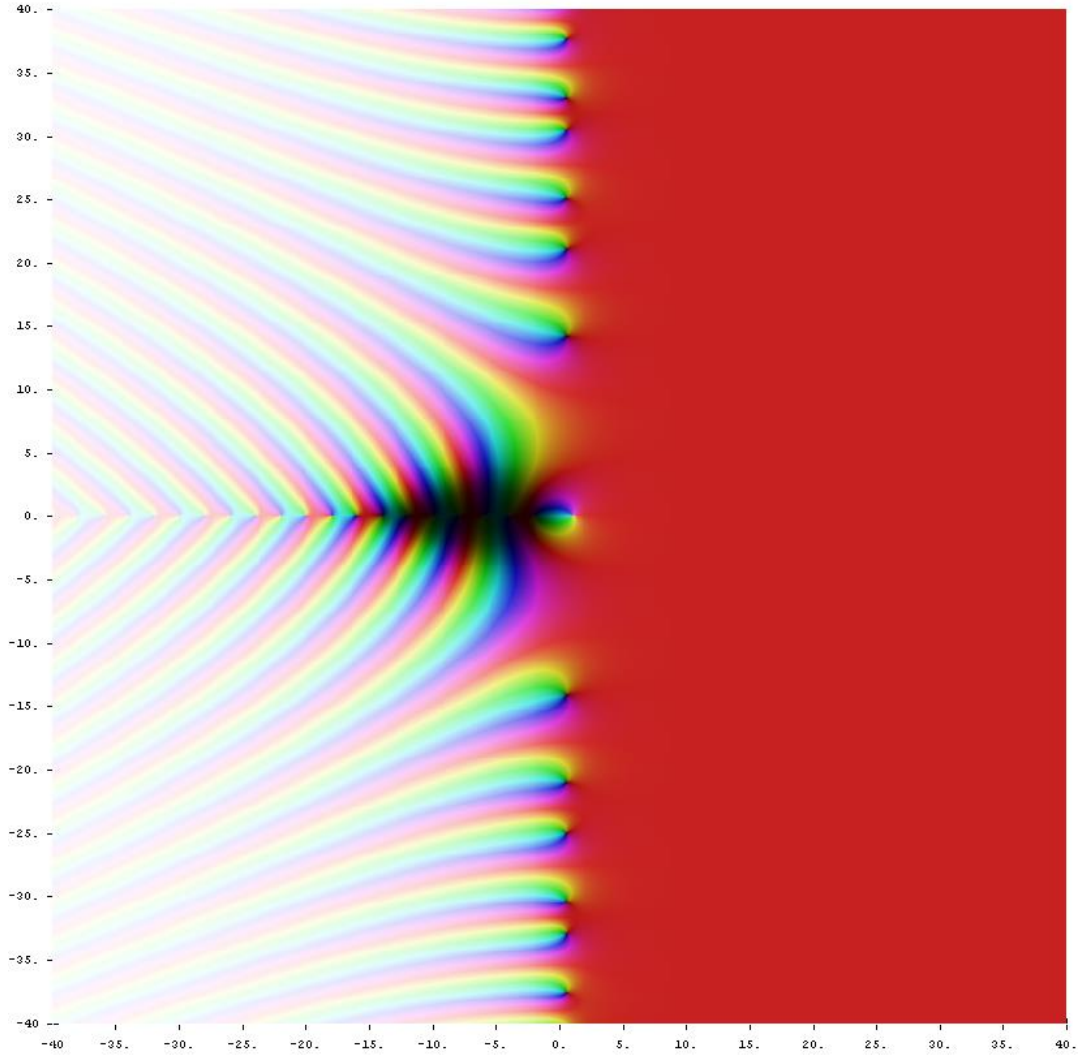
1. If y = f(x), then to **solve** f(x) = 0 means we're trying to ?
2. These are also known as the ? **of the function**.
3. On the graph, these correspond to where the line ?



The roots of a function f are the values x such that $f(x) = 0$.
This corresponds to where its graph crosses the x-axis.

Would you like $1,000,000?

"The roots of a function f are the values x such that $f(x) = 0$."



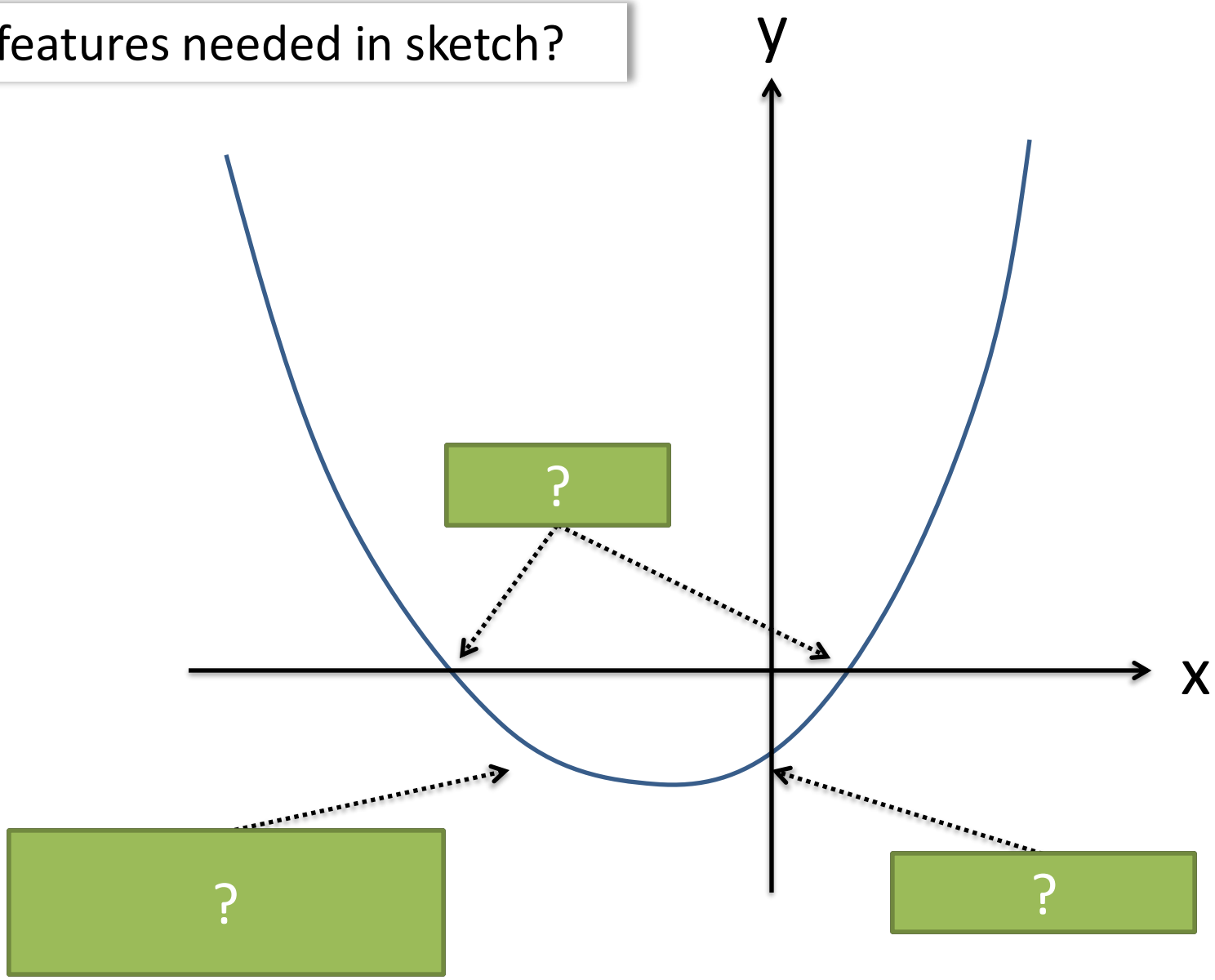
The **Riemann Zeta Function** is a function that allows you to do the infinite sum of powers of reciprocals, e.g.

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$
$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$$

One of the 8 '**Clay Millennium Problems**' (for which solving any attracts a $1,000,000 prize) is to **showing all roots of this function have some particular from**, i.e. the form of x such that $\zeta(x) = 0$.

Sketching Quadratics

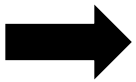
3 features needed in sketch?

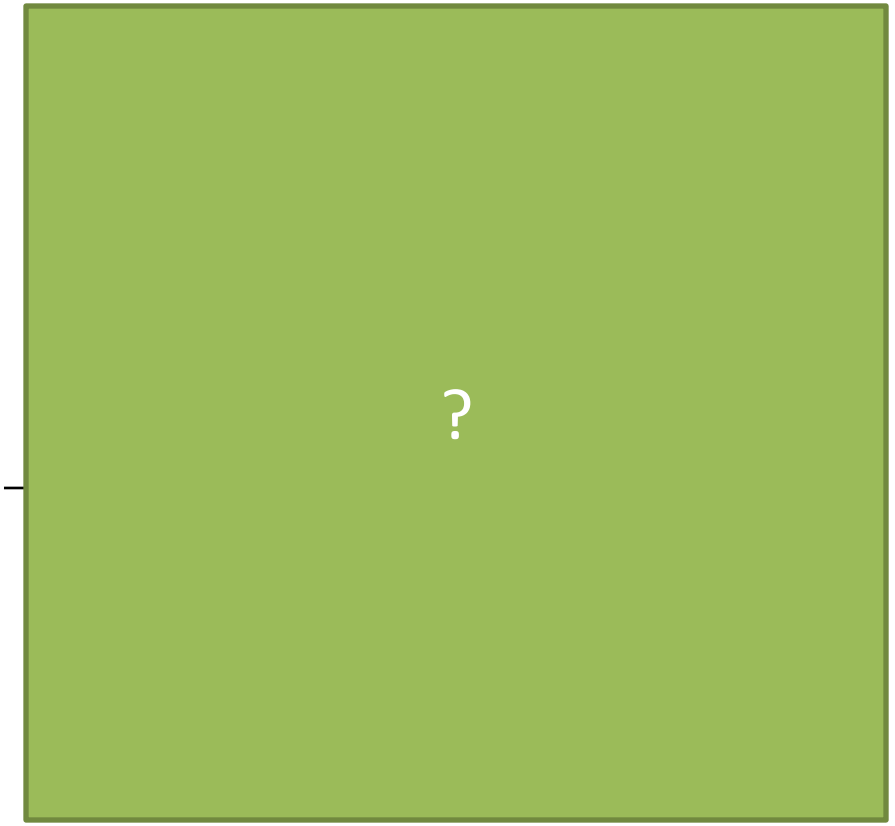


Example 1

1. Roots
2. y-intercept
3. Shape: smiley face or hill?

$$y = x^2 - x - 2$$
$$= (x + 1)(x - 2)$$





So if $y = 0$, i.e. $(x + 1)(x - 2) = 0$, then $x = -1$ or $x = 2$.

When $x = 0$, clearly $y = -2$.

Example 2

1. Roots
2. y-intercept
3. Shape: smiley face or hill?

$$y = -x^2 + 5x - 4$$
$$= \square$$
$$= \square$$
$$= \square$$

?

Bro Tip: We can tidy up by using the minus on the front to swap the order in one of the negations.

?

Example 3

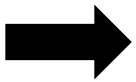
1. Roots
2. y-intercept
3. Shape: smiley face or hill?

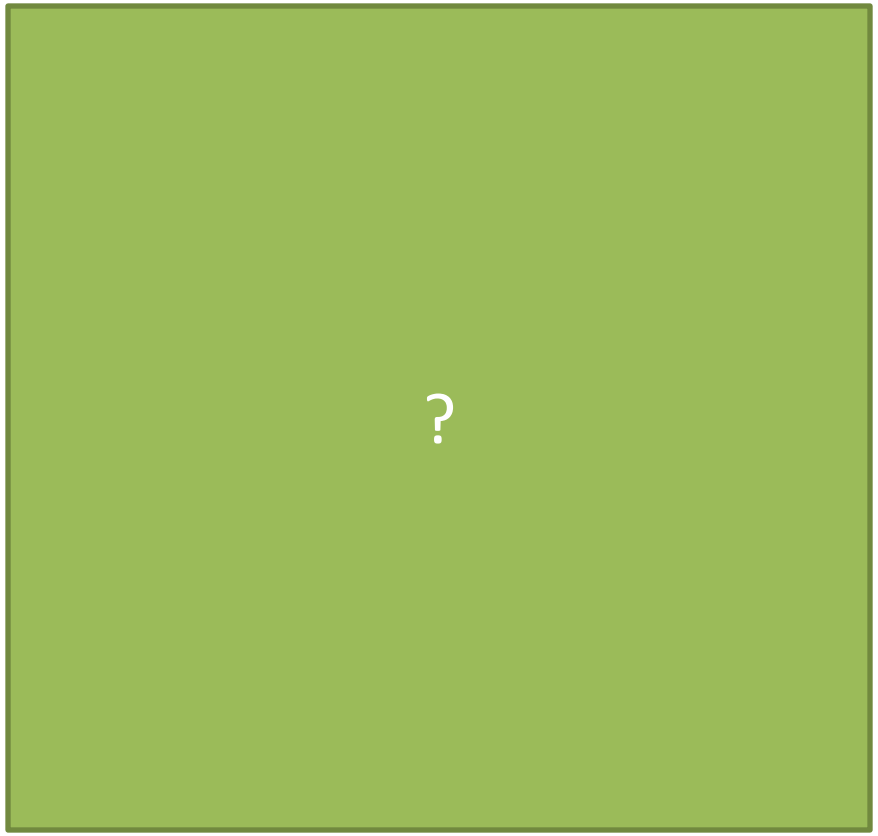
$y = -x^2 + x + 30$

$=$

$=$

$=$





Checking your understanding

$y = x^2 + 3x + 2$

Roots?	?
y-Intercept?	?
∩ or ∪ shape?	?

?

$y = -x^2 + 2x + 8$

Roots?	?
y-Intercept?	?
∩ or ∪ shape?	?

?

Checking your understanding

$y = 9 - x^2$

Roots?	?
y-Intercept?	?
∩ or ∪ shape?	?

?

$y = 2x^2 - 5x - 3$

Roots?	?
y-Intercept?	?
∩ or ∪ shape?	?

?

Exercises

Sketch the following, ensuring you indicate where the curve intercepts either of the axes.

1 $y = x^2 + x - 2$

2 $y = x^2 - 10x + 24$

3 $y = x^2 - 4x - 5$

4 $y = x^2 - 2$

5 $y = 5 - x^2$

6 $y = 10 + 3x - x^2$

7 $y = 9x - 4 - 2x^2$

Determining Min/Max Points

Try to sketch $y = x^2 - 6x + 10$. What problem do you encounter?

?

Completing the square allows us to find where the minimum or maximum point on the graph is…

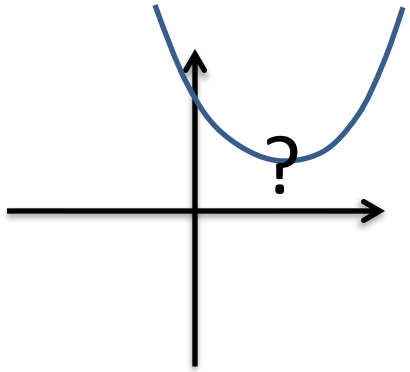
Suppose we complete the square...

$$y = x^2 - 6x + 10$$

$$= \boxed{\quad ? \quad}$$

How could we use this completed square to find the minimum point of the graph?

(Hint: how do you make y as small as possible in this equation?)



?

Write down

When we have a quadratic in the form:

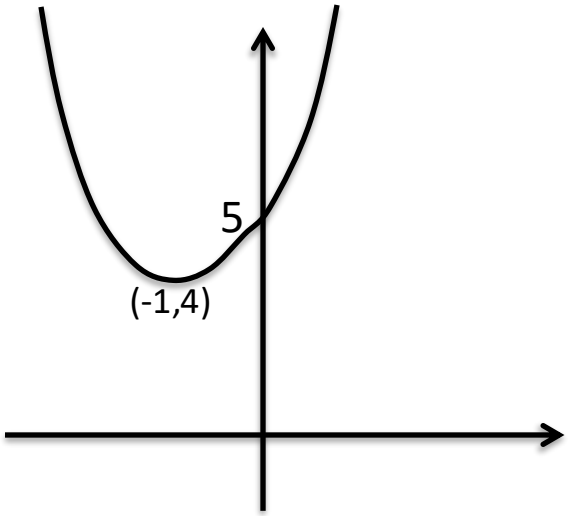
$$y = (x + a)^2 + b$$

The minimum point is $(-a, b)$.

Complete the table, and hence sketch the graphs

	Equation	Completed Square	x at graph min	y at graph min	y-intercept	Roots?
1	$y = x^2 + 2x + 5$	$y = (x + 1)^2 + 4$	-1	4	5	None
2	$y = x^2 - 4x + 7$	?	?	?	?	?
3	$y = x^2 + 6x - 27$	?	?	?	?	?

1



2

?

3

?

Answers to Min/Max Point Card Sort

$y = x^2 - 2x + 3$	?	?
$y = x^2 + 4x + 3$	?	?
$y = -x^2 + 3$	?	?

$y = x^2 - 4x + 3$	?	?
$y = x^2 + 2x + 3$	?	?
$y = x^2 + 2x - 3$	?	?

Quadratic With Maximum Points

If the coefficient of x^2 is negative, we will have a maximum point.

$$y = -x^2 + 8x - 5$$
$$=$$
$$=$$
$$=$$

Completed Square ?

Graph ?

Identifying the maximum point is a similar principle. To <u>maximise</u> y, we want to subtract a small a value as possible. $(x-4)^2$ is smallest when $x = 4$ to make it 0.

Test Your Understanding

Sketch the following, indicating the maximum point and y-intercept.

$y = -x^2 - 10x + 3$

$=$

$=$ Completed Square ?

$=$

Graph ?

$y = -3x^2 + 12x - 6$

$=$

$=$ Completed Square ?

$=$

Graph ?

Exercises

Sketch the following, including the minimum and maximum point (and any intercepts with the axes).

1 $y = x^2 + 8x + 20$

?

2 $y = -x^2 + 4x - 3$

?

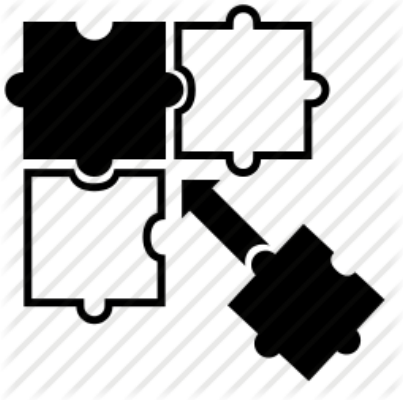
3 $y = 5x^2 - 10x + 10$

?

4 $y = 16x - 4x^2 - 8$

?

For each question...



1. Solve It

Answer the question given, making sure you show working.

2. Swap book

Swap books with your neighbour. They will mark the question according to the provided mark scheme.

3. Show card

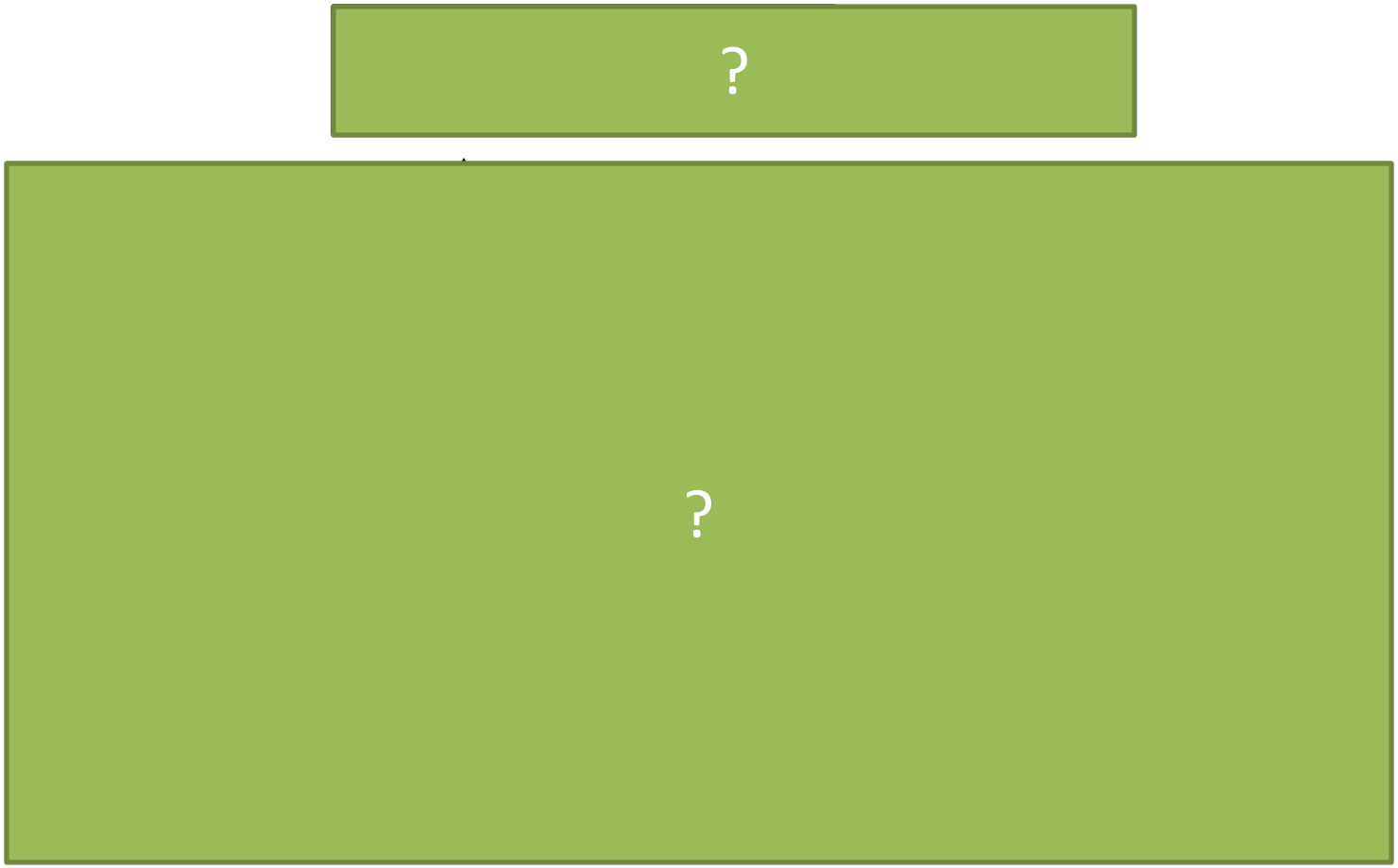
Your neighbour will show:

- Red if all wrong.
- Yellow if partial marks.
- Green if fully correct.

Q1

Put $3x^2 + 24x + 6$ in the form $a(x + p)^2 + q$.

Hence sketch $y = 3x^2 + 24x + 6$



Q2

Put $4x^2 - 6x + 2$ in the form $a(x + p)^2 + q$.

Hence sketch $y = 4x^2 - 6x + 2$

?

(2 marks)

?

Q3

Put $-5x^2 + 10x - 6$ in the form $a(x + p)^2 + q$.

Hence sketch $y = -5x^2 + 10x - 6$

?

(2 marks)

?

Sketching Quadratics

Solving Quadratics By Sketching Graphs

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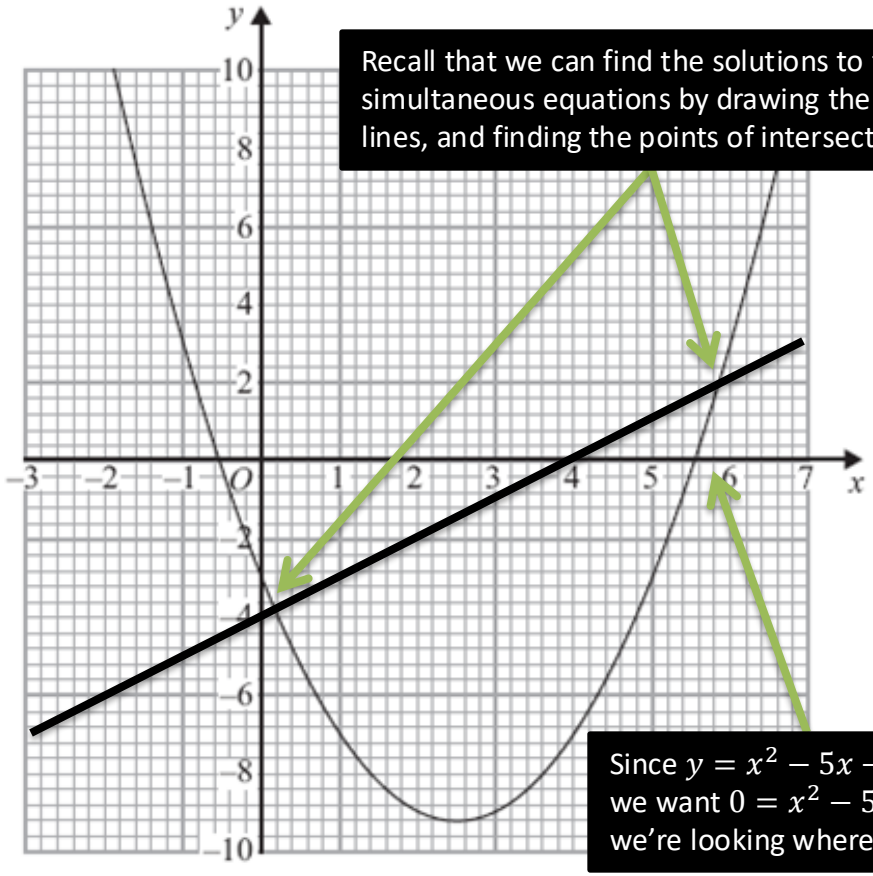
RECAP: Solving Quadratics by using a Graph

Edexcel Nov 2011 NonCalc

The diagram shows the graph of $y = x^2 - 5x - 3$



Recall that we can find the solutions to two simultaneous equations by drawing the two lines, and finding the points of intersection.

Since $y = x^2 - 5x - 3$ and we want $0 = x^2 - 5x - 3$, we're looking where $y = 0$.

a) Use the graph to find estimates for the solutions of

i. $x^2 - 5x - 3 = 0$

?

ii. $x^2 - 5x - 3 = 6$

?

b) Use the graph to find estimates for the solutions of the simultaneous equations:

$y = x^2 - 5x - 3$

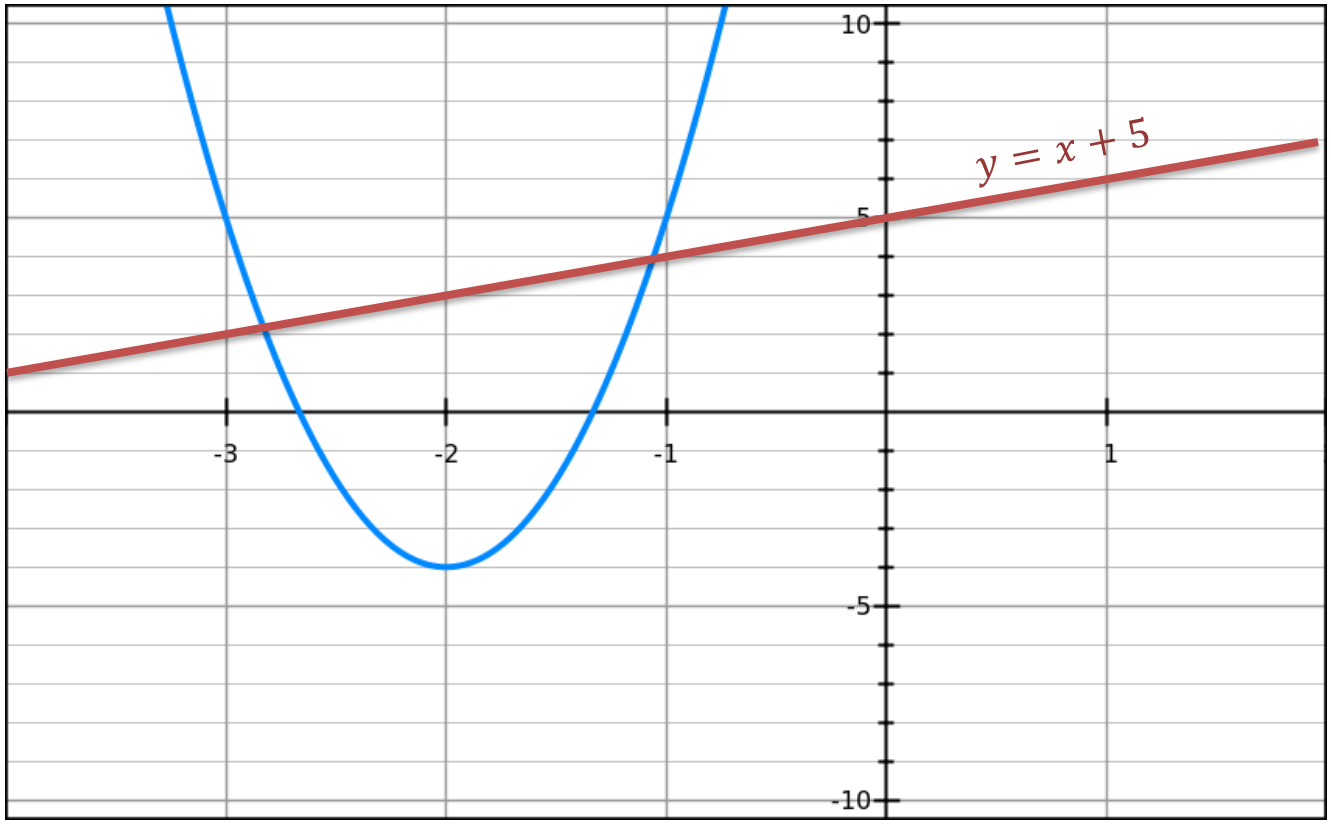
$y = x - 4$

?

Bro Tip: Remember that the easiest way to sketch lines like $y = x - 4$ is to just pick two sensible values of x (e.g. 0 and 4), and see what y is for each. Then join up the two points with a line.

Test Your Understanding

(see supplied sheet)



The graph shows $y = 9x^2 + 36x + 32$.

a Estimate the solutions to $9x^2 + 36x + 32 = 0$

?

b Determine the solutions to $9x^2 + 36x + 32 = 5$

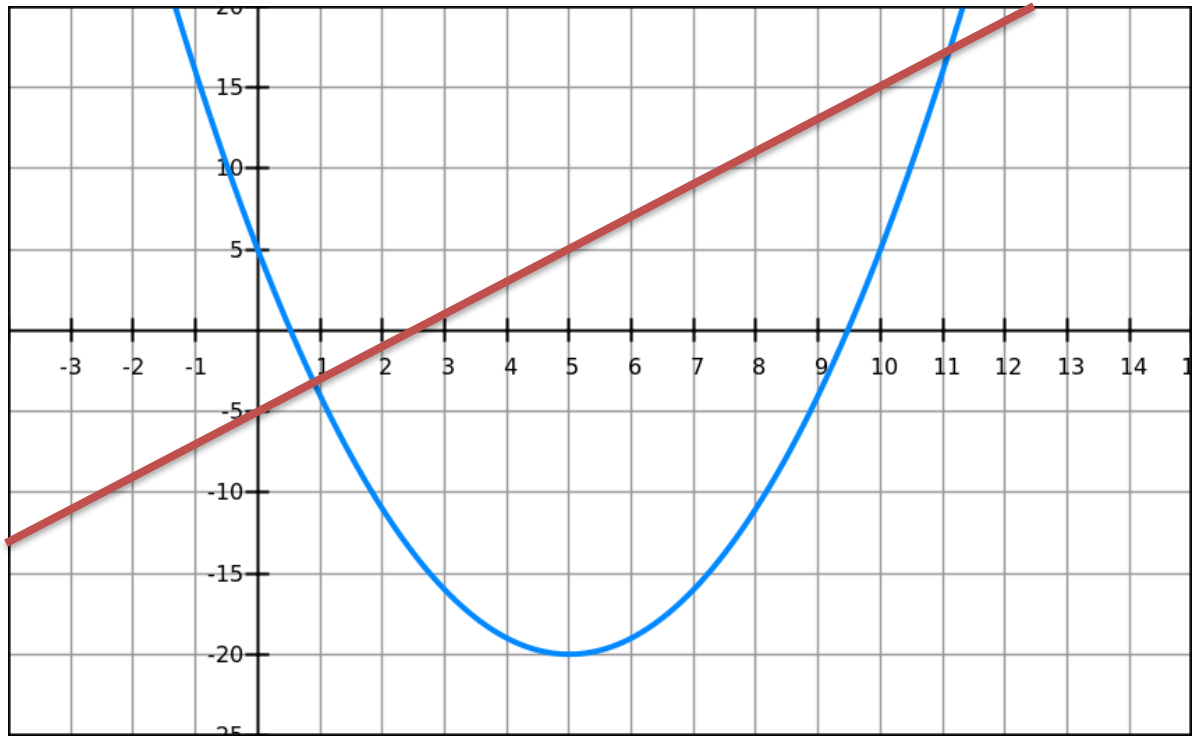
?

c By using your graph, solve the simultaneous equations:

$$y = 9x^2 + 36x + 32$$
$$y = x + 5$$

?

Question 1



The above is the graph $y = x^2 - 10x + 5$.

a Estimate the solutions of $x^2 - 10x + 5 = 0$?

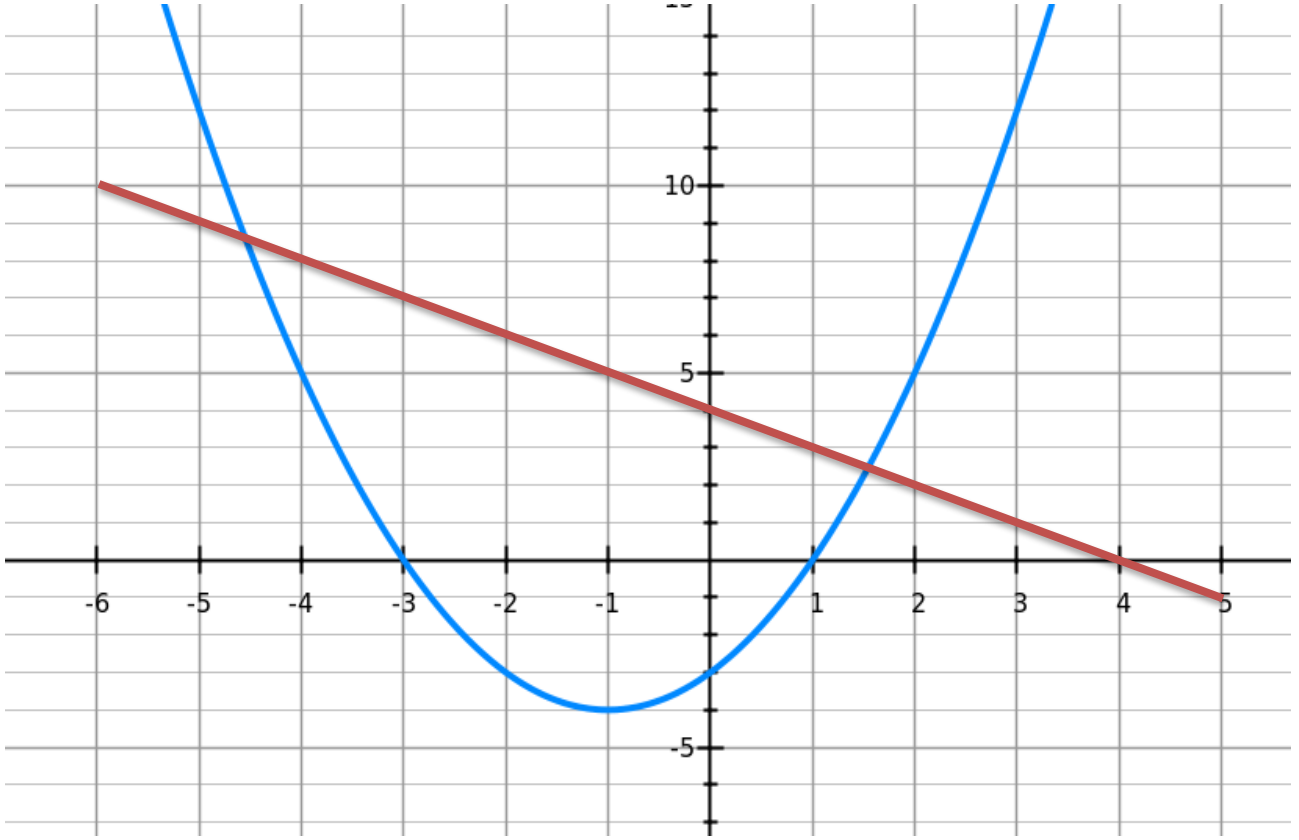
b Estimate the solutions of $x^2 - 10x + 5 = -10$?

c By drawing an appropriate graph, estimate the solutions to the simultaneous equations:

$$y = x^2 - 10x + 5$$
$$y = 2x - 5$$

?

Question 2



What are the roots of $f(x)$?

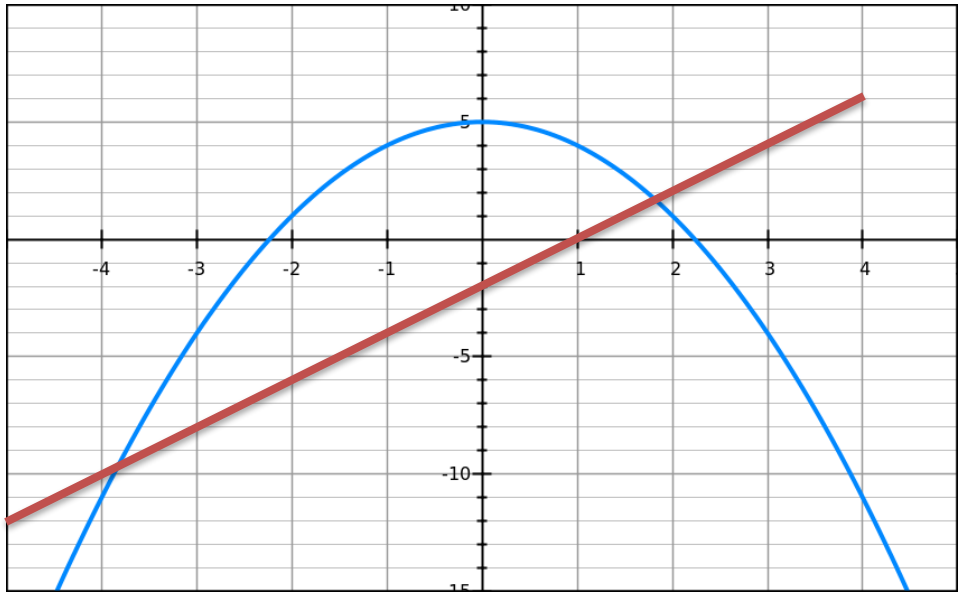
?

By drawing a s… solutions to $x^2 + 2x - 3 = 4 - x$

?

Question 3

Use the grid provided to draw the graph of $y = 5 - x^2$ (Hint: use an appropriate table of values to work out the points first)



Use your graph to find an estimate of the solution to $5 - x^2 = 3$

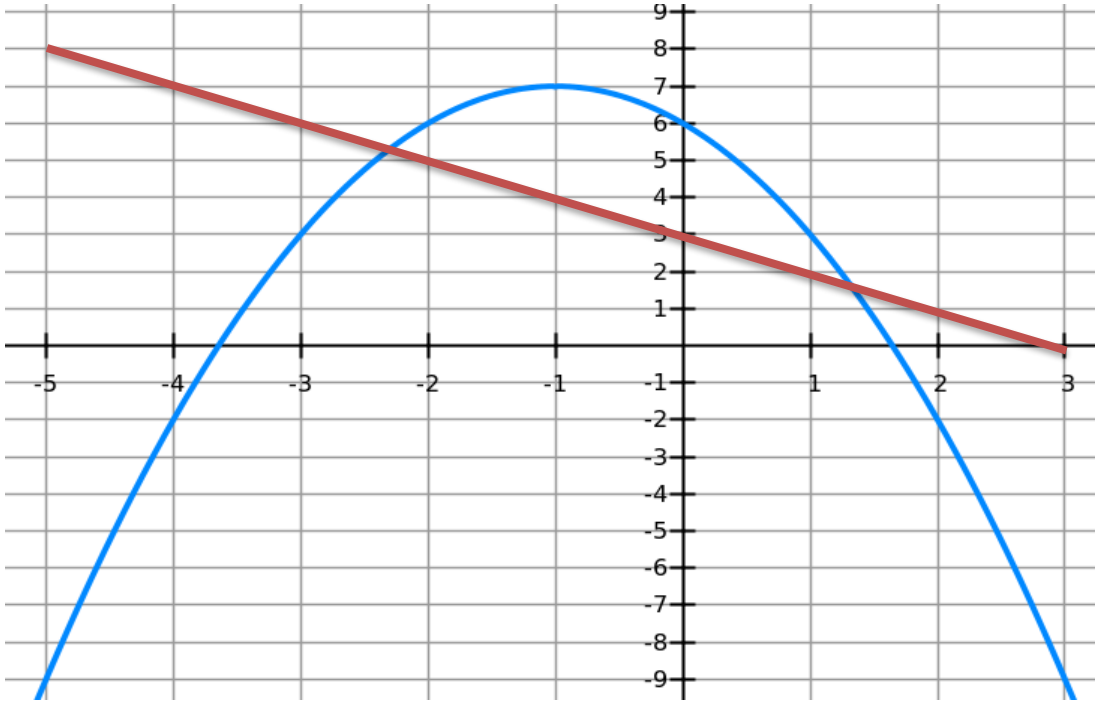
?

By sketching a suitable lin us equations:

$$y = 5 - x^2$$
$$y = 2x - 2$$
$$\boldsymbol{x = -3.8, y = -9.7}$$

?

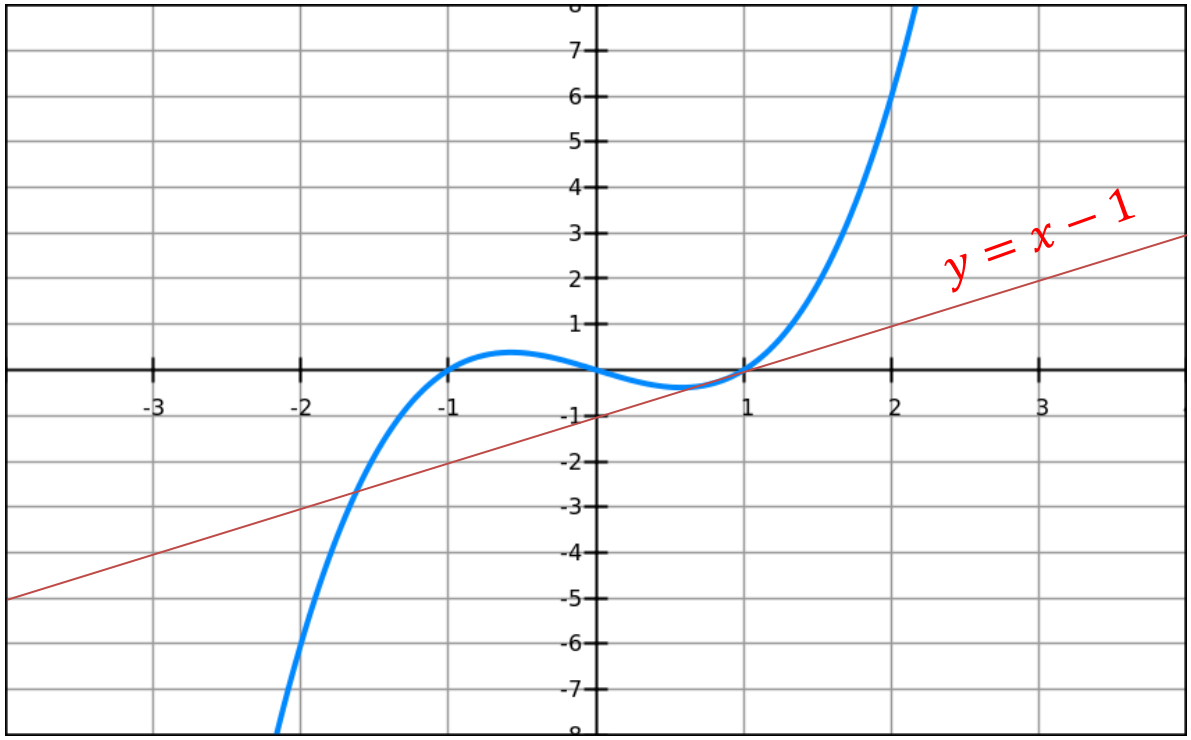
Question 4



The sketch below is of $y = 6 - x^2 - 2x$. By sketching a suitable graph, approximate the solutions to $-x^2 - x + 3 = 0$ (Hint: Manipulate this to put $6 - x^2 - 2x$ on one side of the equation and see what's on the other side of the equation).

?

Question 5



Above is a sketch of the cubic $y = x^3 - x$. By sketching an appropriate line, estimate solutions to the equation $x^3 - 2x + 1 = 0$.

?

TRUE OR FALSE

The axis of symmetry passes through the vertex of the parabola.

TRUE

TRUE OR FALSE

If the parabola opens downward, the vertex represents the minimum value.

FALSE

TRUE OR FALSE

If the parabola opens downward, the vertex represents the minimum value.

FALSE

Question 1: Rectangle Area (Classic)

A rectangle has a **fixed perimeter of 20 cm**.

1. Express the **area** of the rectangle as a function of one side.
2. Find the dimensions that **maximize the area**.

Question 3: Quadratic Profit Problem (Advanced)

A company produces x units of a product. The **profit** in £ is given by:

$$P(x) = -2x^2 + 120x - 100$$

Question 1: Rectangle Area

- Let one side = x, other side = $10 - x$ (perimeter = 20)

$$A(x) = x(10 - x) = 10x - x^2$$

- Maximum occurs at vertex: $x = -\frac{b}{2a} = -\frac{10}{-2} = 5$
- **Dimensions:** 5 cm × 5 cm → **maximum area = 25 cm²**

Question 3: Quadratic Profit

$$P(x) = -2x^2 + 120x - 100$$

- Maximum occurs at vertex: $x = -\frac{b}{2a} = -\frac{120}{-4} = 30$
- Maximum profit:

$$P(30) = -2(30)^2 + 120(30) - 100 = 1700$$

- Positive profit: solve $P(x) > 0$

$$-2x^2 + 120x - 100 > 0 \Rightarrow x^2 - 60x + 50 < 0$$

- Roots: $x = \frac{60 \pm \sqrt{3600 - 200}}{2} = \frac{60 \pm \sqrt{3400}}{2} \approx \frac{60 \pm 58.31}{2}$

$$x \in (0.85, 59.15)$$