UNIT 3 KNOWLEDGE - CALCULUS 12 – DIFFERENTIATION: COMPOSITE, IMPLICIT, & INVERSE FUNCTIONS

- **3.1** • The Chain Rule ✓
- **3.2** • Implicit Differentiation ✓
- **3.3** • Differentiating Inverse Functions
- **3.4** Differentiating Inverse Trigonometric Functions
- **3.5** • Selecting Procedures for Calculating Derivatives
- **3.6** • Calculating Higher-Order Derivatives

What Will We Learn?

- What is an inverse function?

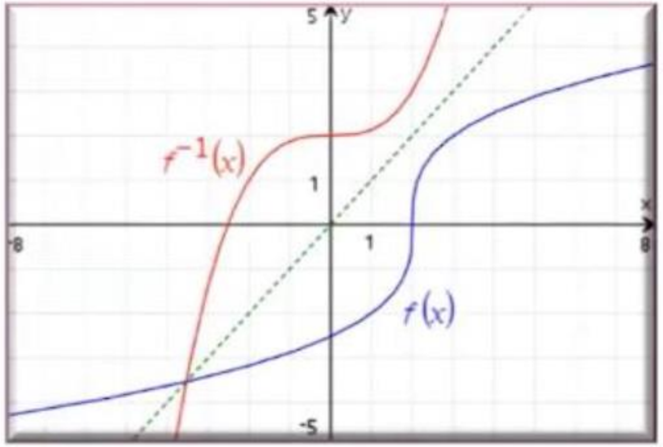
- What are the three ways in which the derivative of an inverse function can be calculated?

Key Terms: Inverse Function Recap from Algebra 2/Precalculus

Definition: An inverse function is a function that "undoes" or "reverses" another function. A function g is the inverse of function f if whenever $y = f(x)$ then $x = g(y)$.

Algebraic Relationship: The inverse of a function can be found by swapping the x and y values within that function. If the functions f and g are inverses of one another, then $f(g(x))=x$ and $g(f(x))=x$.

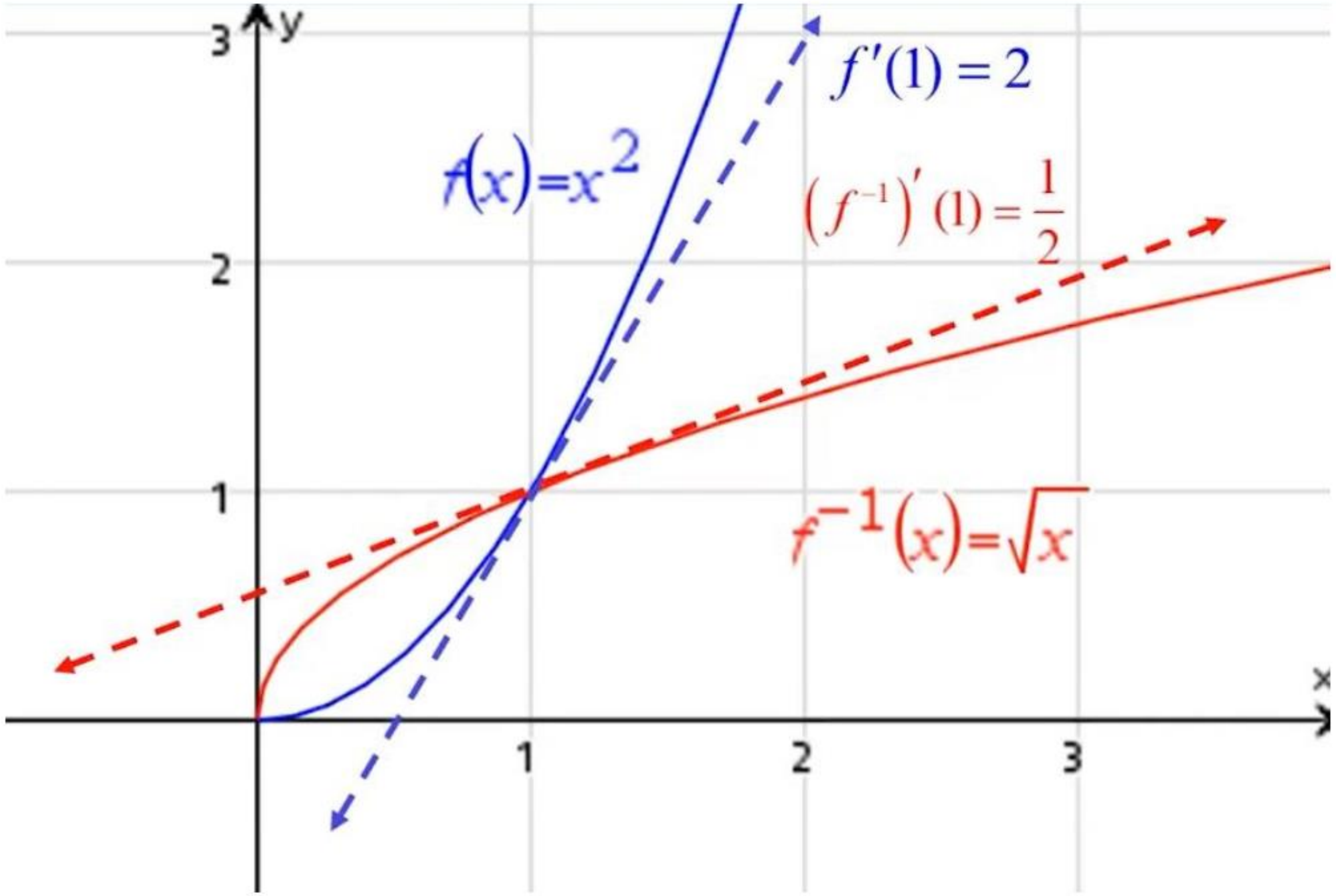
Graphical Relationship: The graphs of two inverse functions are symmetrical with respect to the line $y = x$.



Example 1- If $f(x) = x^2 \ for \ x \geq 0$, find the derivative of $f^{-1}(x) \ at \ x = 1$.

Steps using Method 1	Solution
$f^{-1}(x) \ at \ x = 1.$	This means f-inverse derivative at 1 $\therefore find \quad (f^{-1})' \, (1)$
Step 1: Switch $y \ and \ x$ & solve for y by taking root	$y = x^2 \rightarrow x = y^2$ $\sqrt{x} = \sqrt{y^2}$
Step 2: y is the inverse which is now the root of x Since $x \geq 0$ in the original function $f(x), y \geq 0 \ in \ f^{-1}(x)$	$y = f^{-1}(x) = \sqrt{x}$
Step 3: Take derivative & solve for $x = 1$	$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ $(f^{-1})'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$

Example 1- If $f(x) = x^2 \; for \; x \geq 0$, find the derivative of $f^{-1}(x) \; at \; x = 1$.



Graphical depiction of the function and its inverse form.

- When $f(x) \geq 0$, the tangent line(blue) when $x = 1$ has a slope of 2
- BUT the derivative of the inverse (slope of the red line) is $\frac{1}{2}$ **(reciprocal of 2)**

Example 2- If $f(x) = \frac{1}{2}x^3 - x + 3$, find the derivative of $f^{-1}(x)\ at\ x = 5$.

Steps using Method 2	**Solution**
$f^{-1}(x)\ at\ x = 5.$	This means f-inverse derivative at 5 $\therefore find\ \ (f^{-1})'(5)$
Step 1: Switch $y\ and\ x$	$y = \frac{1}{2}x^3 - x + 3$ $x = \frac{1}{2}y^3 - y + 3$
Step 2: Use implicit differentiation	$\frac{d}{dx}[x] = \frac{d}{dx}\left[\frac{1}{2}y^3 - y + 3\ \right]$ $1 = \left[\frac{3}{2}y^2 \cdot \frac{dy}{dx} - \frac{dy}{dx} + 0\ \right]$ $1 = \frac{dy}{dx}\left[\frac{3}{2}y^2 - 1\ \right]$ $\frac{dy}{dx} = \frac{1}{\left(\frac{3}{2}y^2 - 1\right)}$

Example 2- If $f(x) = \frac{1}{2}x^3 - x + 3$, find the derivative of $f^{-1}(x)\ at\ x = 5$.

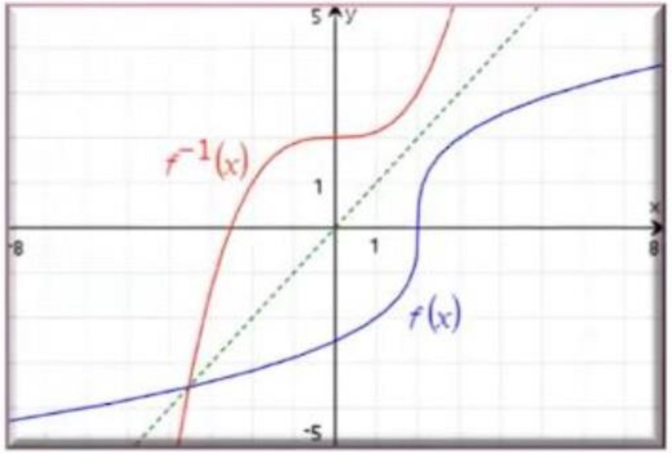
Steps using Method 2	Solution
Step 2: Use implicit differentiation	$\frac{dy}{dx} = \frac{1}{\left(\frac{3}{2}y^2 - 1\right)}$
What do we plug in for y?	Recall our inverse, $x = \frac{1}{2}y^3 - y + 3$
Step 3: plug in 5 for x & solve for y	$5 = \frac{1}{2}y^3 - y + 3$
Use "trial and error" to find y Try 1 (LHS = RHS?)	$5 \neq \frac{1}{2}(1)^3 - (1) + 3$
Try 2 (LHS = RHS?)	$5 = \frac{1}{2}(2)^3 - (2) + 3, \qquad so\ y = 2$
	So, $\left(f^{-1}\right)'(5) = \left.\frac{dy}{dx}\right\|_{y=2} = \frac{1}{\left(\frac{3}{2}(2)^2 - 1\right)} = \frac{1}{5}$

Key Takeaways

Definition: An inverse function is a function that "undoes" or "reverses" another function. A function g is the inverse of function f if whenever $y = f(x)$ then $x = g(y)$.

Algebraic Relationship: The inverse of a function can be found by swapping the x and y values within that function. If the functions f and g are inverses of one another, then $f(g(x))=x$ and $g(f(x))=x$.

Graphical Relationship: The graphs of two inverse functions are symmetrical with respect to the line $y = x$.



Next-> What Will We Learn?

- We'll focus on a method that works best in all situations when taking the derivative of an inverse function.

Example 2: If $f(x) = \frac{1}{2}x^3 - x + 3$, find the derivative of $f^{-1}(x)$ at $x = 5$.

$$\frac{dy}{dx} = \frac{1}{\left(\frac{3}{2}y^2 - 1\right)}$$

So, $\left(f^{-1}\right)'(5) = \left.\frac{dy}{dx}\right|_{y=2} = \frac{1}{\left(\frac{3}{2}(2)^2 - 1\right)} = \boxed{\frac{1}{5}}$

N.B. The derivative of the top is equivalent to the bottom except for the variables x and y switched.

<u>This relationship will lead us to Method 3.</u>

Method 3: Derivatives of Inverse Functions: A Formulaic Approach

The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I. If f has an inverse function f^{-1}, then f^{-1} is differentiable at any x for which $f'\left(f^{-1}(x)\right) \neq 0$ and

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Helpful Mnemonic: I.D.D.I. $\quad (\textbf{inverse})\boldsymbol{derivative}(\boldsymbol{x}) = \dfrac{\mathbf{1}}{\boldsymbol{derivative} \cdot \boldsymbol{inverse}(\boldsymbol{x})}$

Example 3- Selected values of a strictly monotonic function $g(x)$ and its derivative $g'(x)$ are shown in the table below.

x	-3	-1	1	4
$g(x)$	5	1	0	-3
$g'(x)$	-4	$-\frac{1}{5}$	$-\frac{1}{6}$	-2

$$Find \quad (g^{-1})'(-3)$$

A monotonic function is **a function which is either entirely non-increasing or non-decreasing**.

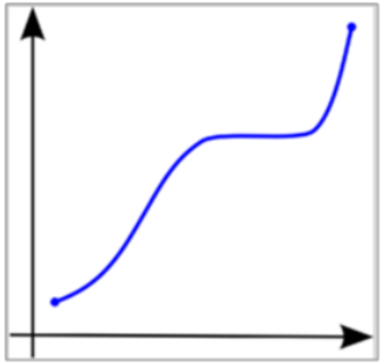


Figure 1 - A monotonically increasing function

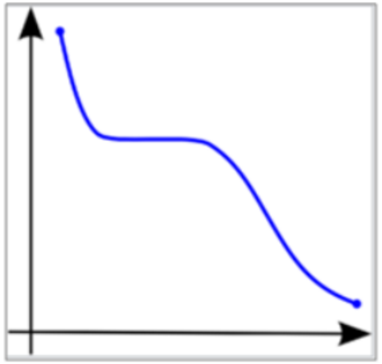


Figure 2 - A monotonically decreasing function

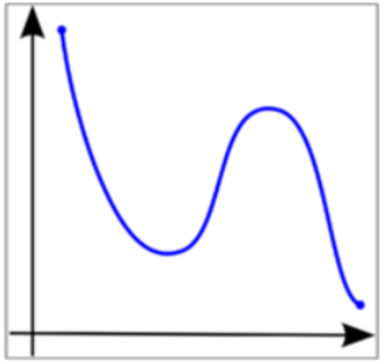


Figure 3 - A function that is not monotonic

Example 3- Selected values of a strictly monotonic function $g(x)$ and its derivative $g'(x)$ are shown in the table below.

x	-3	-1	1	4
$g(x)$	5	1	0	-3
$g'(x)$	-4	$-\frac{1}{5}$	$-\frac{1}{6}$	-2

$$Find \quad (g^{-1})'(-3)$$

Steps using Method 3	**Solution**
Step 1: Look across the g-row and find the $g - value$ that is equivalent to -3. At that $g - value$, what's the x-value?	$Find \ (g^{-1})'(-3)$ $(g^{-1})'(x) = \dfrac{1}{g'(g^{-1}(x))}$ $g^{-1}(-3) = \mathbf{4}$
Step 2: At that $g - value$, plug in the x-value there and solve	$\therefore (g^{-1})'(-3) = \dfrac{1}{g'(\mathbf{4})}$ $= \dfrac{1}{-2} = -\dfrac{\mathbf{1}}{\mathbf{2}}$

Example 4- Selected values of a strictly monotonic function $g(x)$ and its derivative $g'(x)$ are shown in the table below.

x	-3	-1	1	4
$g(x)$	5	1	0	-3
$g'(x)$	-4	$-\frac{1}{5}$	$-\frac{1}{6}$	-2

$$Find \quad (g^{-1})'(1)$$

Steps using Method 3	Solution
Step 1: Look across the g-row and find the $g-value$ that is equivalent to 1. At that $g-value$, what's the x-value?	$Find \quad (g^{-1})'(1)$ $(g^{-1})'(x) = \dfrac{1}{g'(g^{-1}(x))}$ $g^{-1}(1) = -1$
Step 2: At that $g-value$, plug in the x-value there and solve	$\therefore (g^{-1})'(1) = \dfrac{1}{g'(-1)}$ $= \dfrac{1}{-\frac{1}{5}} = -5$

Summary/Key Takeaways

Derivatives of Inverse Functions: A Formulaic Approach

The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I. If f has an inverse function f^{-1}, then f^{-1} is differentiable at any x for which $f'(f^{-1}(x)) \neq 0$ and

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Helpful Mnemonic: I.D.D.I. $\quad (\text{inverse})\,derivative(x) = \dfrac{1}{derivative \cdot inverse(x)}$