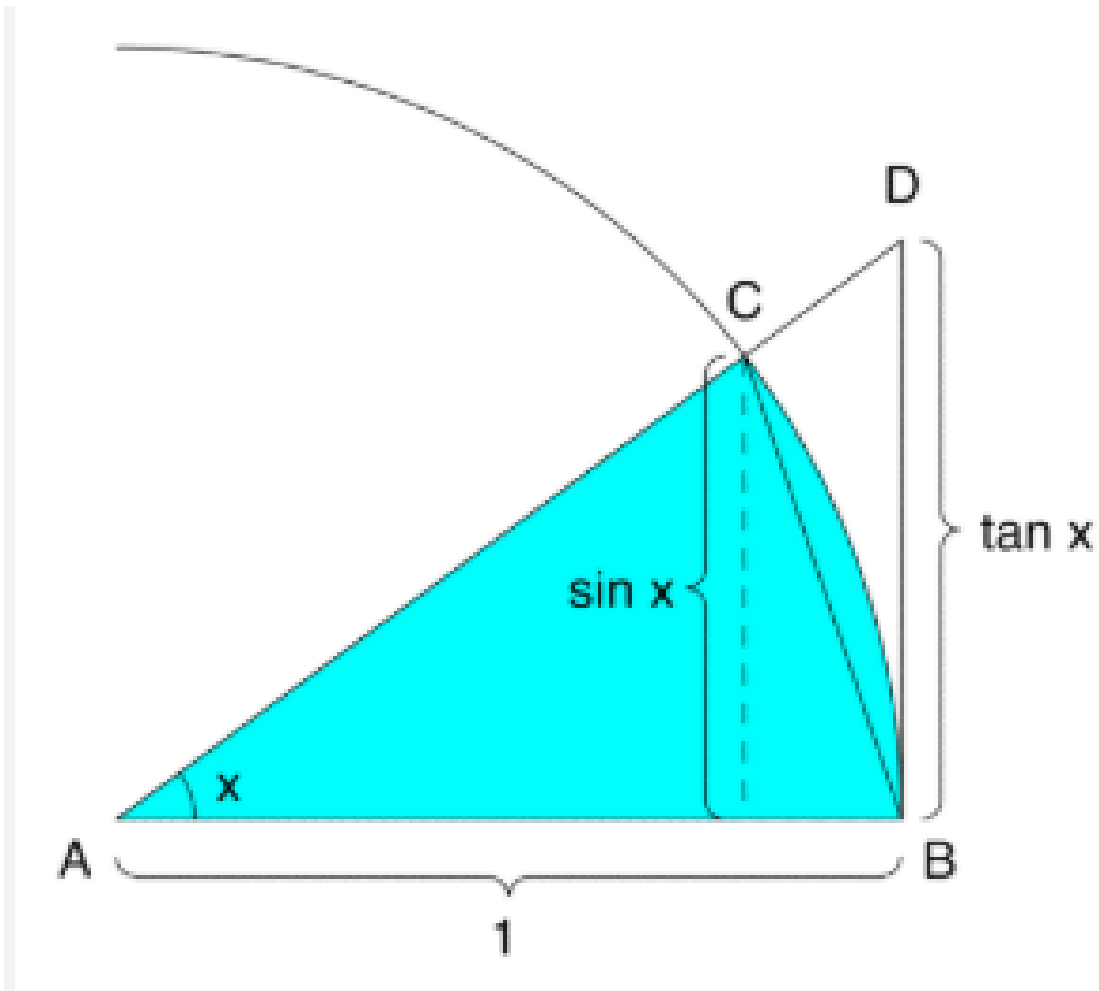


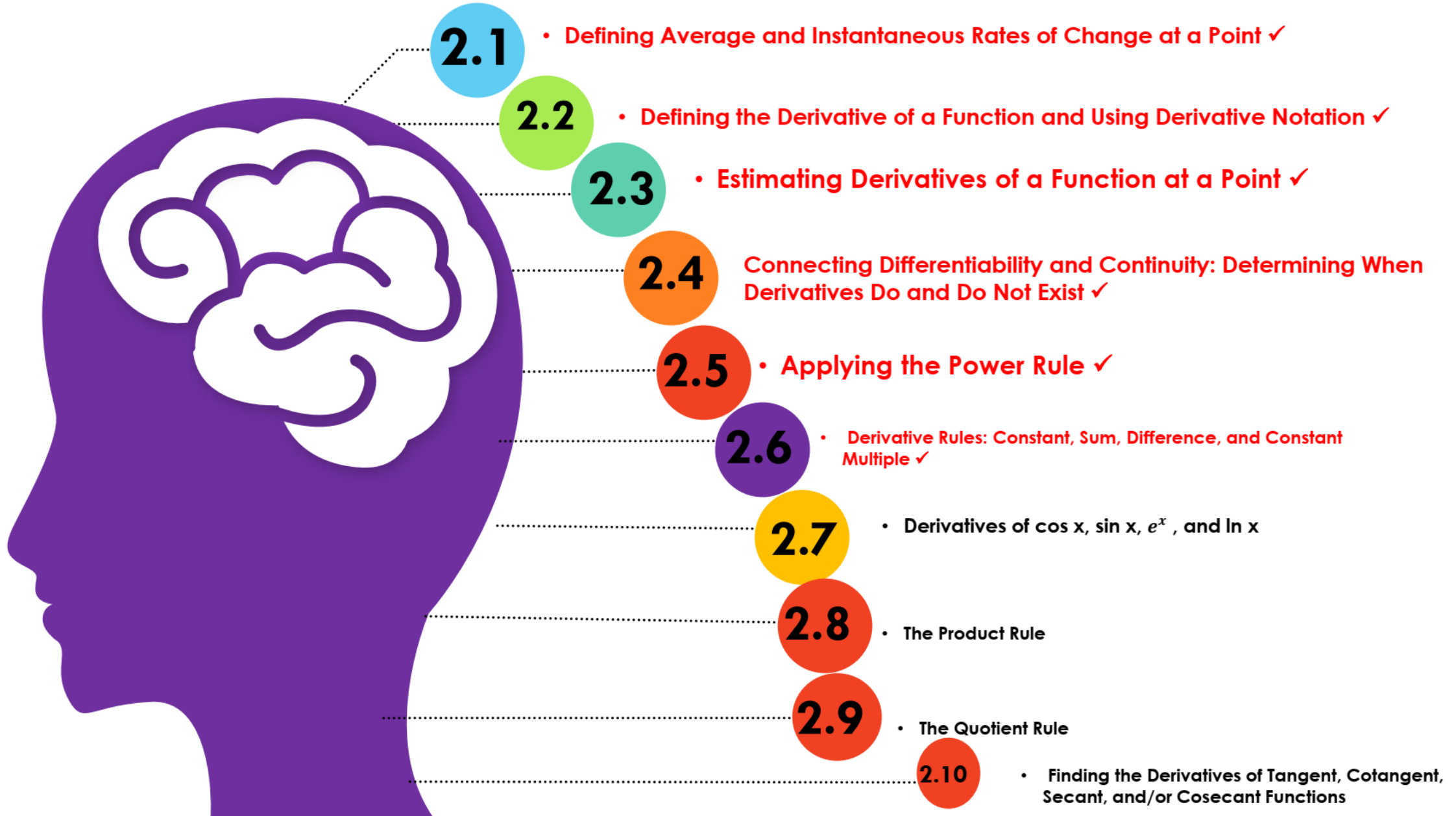
TURN and TALK

Prove using the diagram below and squeeze theorem



$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$$

UNIT 2 KNOWLEDGE - CALCULUS 12 – DIFFERENTIATION: DEFINITION AND FUNDAMENTAL PROPERTIES



What Will We Learn?

- We'll look at the basic rules for differentiating four of the most common transcendental functions - $\cos x$, $\sin x$, e^x , and $\ln x$.

Recall from Precalculus: the Sum-Angle Formulas from Precalculus

ADDITION AND SUBTRACTION FORMULAS

Formulas for sine:

$$\sin(s + t) = \sin s \cos t + \cos s \sin t$$

$$\sin(s - t) = \sin s \cos t - \cos s \sin t$$

Formulas for cosine:

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

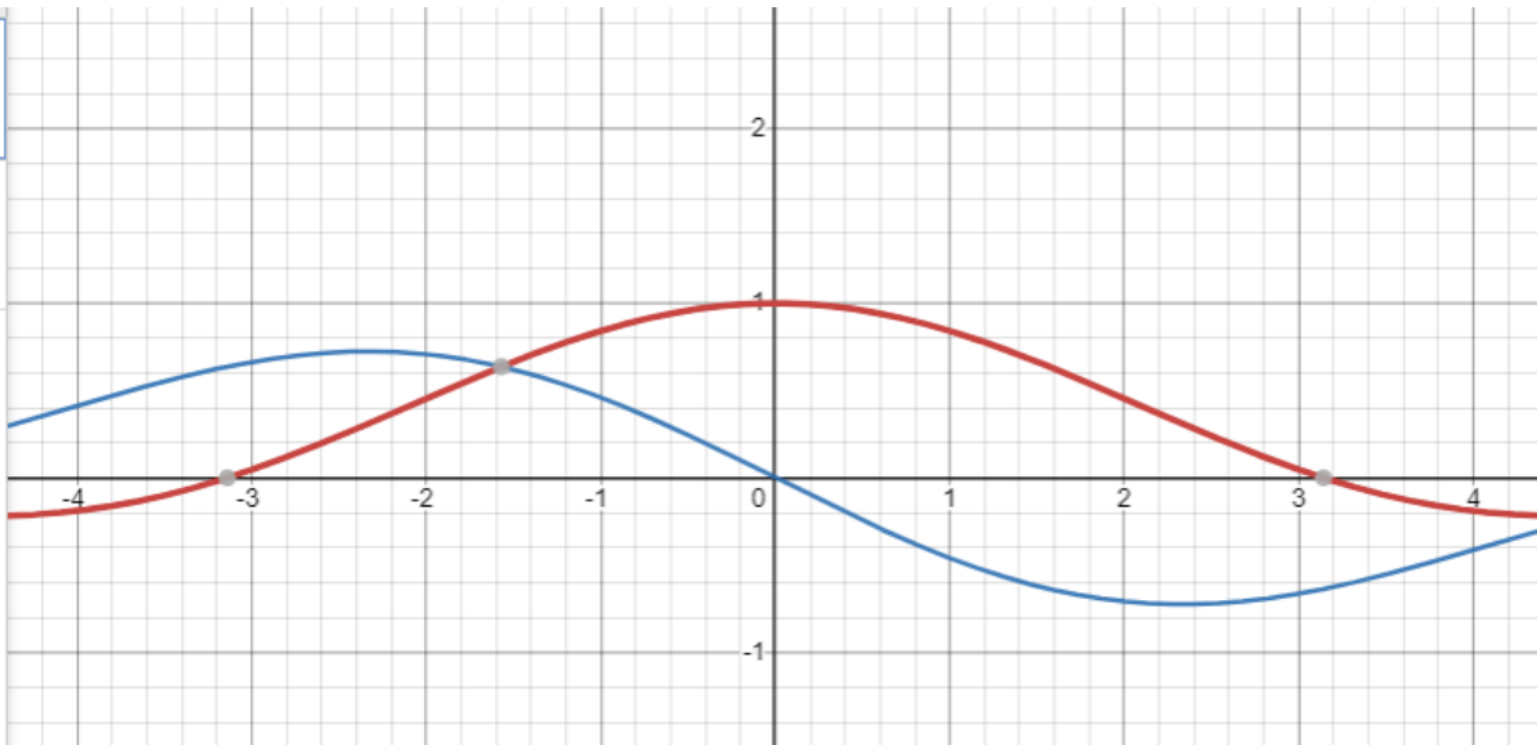
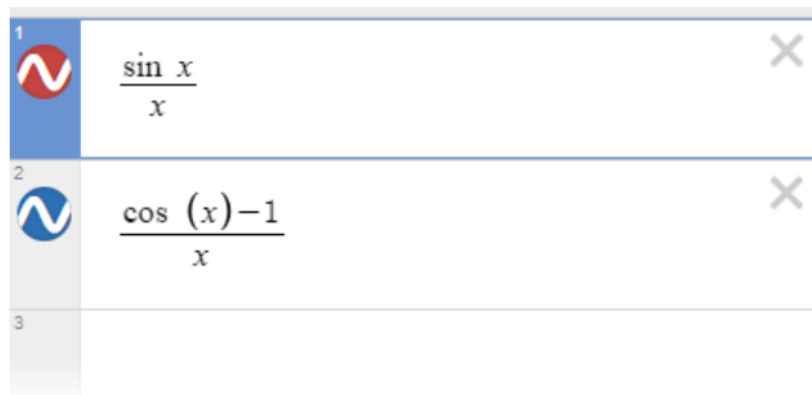
$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$

Formulas for tangent:

$$\tan(s + t) = \frac{\tan s + \tan t}{1 - \tan s \tan t}$$

$$\tan(s - t) = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

Recall: From Unit 1



$$\lim_{h \rightarrow 0} \left(\frac{\cos(h) - 1}{h} \right) = 0$$

Recall from Unit 1

$$\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1$$

Four Most Common Transcendental Derivative Formulas

$$\frac{d}{dx} [\sin x] = \cos x$$

$$\frac{d}{dx} [\cos x] = -\sin x$$

$$\frac{d}{dx} [e^x] = e^x$$

$$\frac{d}{dx} [\ln x] = \frac{1}{x}$$

These four functions are the four most common transcendental functions that you will encounter in calculus!

Let's see a proof!

Proving $\frac{d}{dx} [\cos x] = -\sin x$ Using the Limit Definition of Derivative

Formulas for cosine:

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$

We'll use the limit definition of the derivative and replace it with $\cos x$

$$\frac{d}{dx} [\cos x] = \lim_{h \rightarrow 0} \frac{\cos(x + h) - \cos x}{h}$$

Next, we'll use trigonometric identity (sum-angle identity for cosine)

$$\frac{d}{dx} [\cos x] = \lim_{h \rightarrow 0} \frac{\cos(x) \cdot \cos(h) - \sin(x) \cdot \sin(h) - \cos(x)}{h}$$

Next, we'll put like terms together, factor and apply limit

$$= \lim_{h \rightarrow 0} \frac{\cos(x) \cdot \cos(h) - \cos(x) - \sin(x) \cdot \sin(h)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cos(x) (\cos(h) - 1) - \sin(x) \cdot \sin(h)}{h}$$

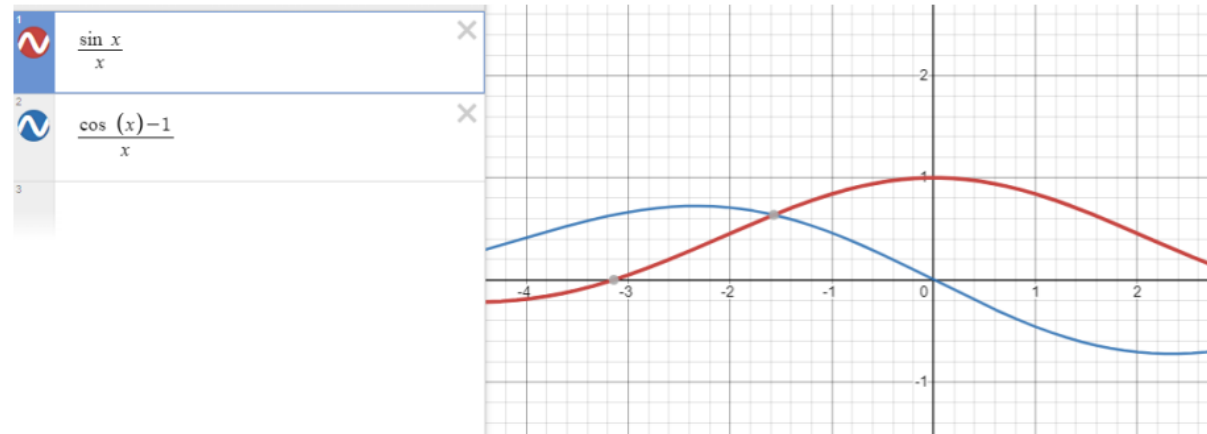
$$= \cos x \cdot \lim_{h \rightarrow 0} \left(\frac{\cos(h) - 1}{h} \right) - \sin x \cdot \lim_{h \rightarrow 0} \frac{\sin(h)}{h}$$

Proving $\frac{d}{dx}[\cos x] = -\sin x$ Using the Limit Definition of Derivative

$$\frac{d}{dx}[\cos x] = \cos x \cdot \lim_{h \rightarrow 0} \left(\frac{\cos(h) - 1}{h} \right) - \sin x \cdot \lim_{h \rightarrow 0} \frac{\sin(h)}{h}$$

$$= \cos x \cdot (0) - \sin x (1)$$

$$= -\sin x$$



Derivative of an exponential function

Let $f(x) = e^x$

Then by definition

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \rightarrow 0} e^x \left(\frac{e^h - 1}{h} \right) \\ &= e^x \lim_{h \rightarrow 0} \frac{(e^h - 1)}{h} = e^x \cdot 1 = e^x \end{aligned}$$

Hence, $\frac{d}{dx} (e^x) = e^x$

The derivative of $\ln x$

To find the derivative of $f(x) = \ln x$, we again recall the definition of Euler's number e :

$$\lim_{h \rightarrow 0} (1+h)^{1/h} = e$$

where e is a non-fractional number roughly equal to 2.718281828459045235.

Note that $\ln x$ is the inverse of the exponential function e^x . Thus $\ln e^x = x$ for all x . In particular,

$$\ln e = 1.$$

Let $f(x) = \ln x$. Then

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\ln(x+h) - \ln x}{h} \\ &= \frac{\ln\left(\frac{x+h}{x}\right)}{h} \\ &= \frac{1}{h} \ln\left(1 + \frac{h}{x}\right) \end{aligned}$$

Let $\frac{h}{x} = k$. Thus $h = kx$ and the above becomes

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{1}{h} \ln\left(1 + \frac{h}{x}\right) \\ &= \frac{1}{kx} \ln(1+k) \\ &= \frac{1}{x} \ln(1+k)^{1/k} \end{aligned}$$

As $h \rightarrow 0$, $k \rightarrow 0$. Hence by the definition of e , from the above we have

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \frac{1}{x} \ln e = \frac{1}{x}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Example: Find the derivative of each of the following.

a) $f(x) = 5e^x$



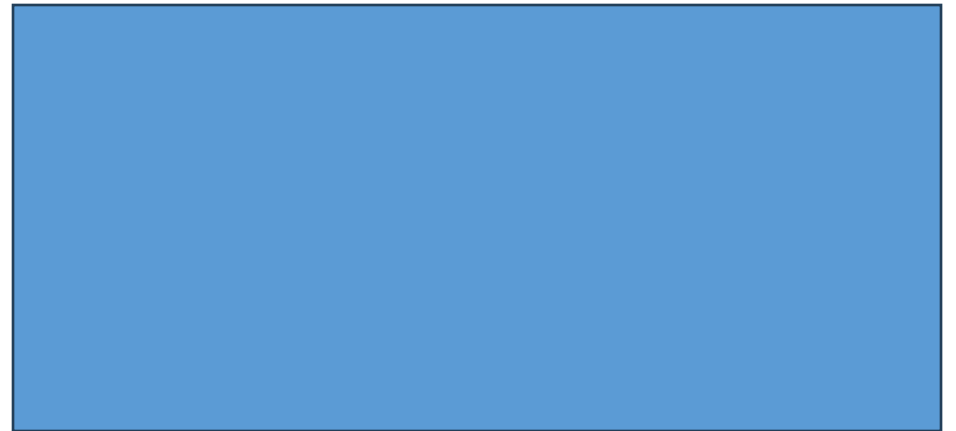
b) $g(x) = e^x - 4x$



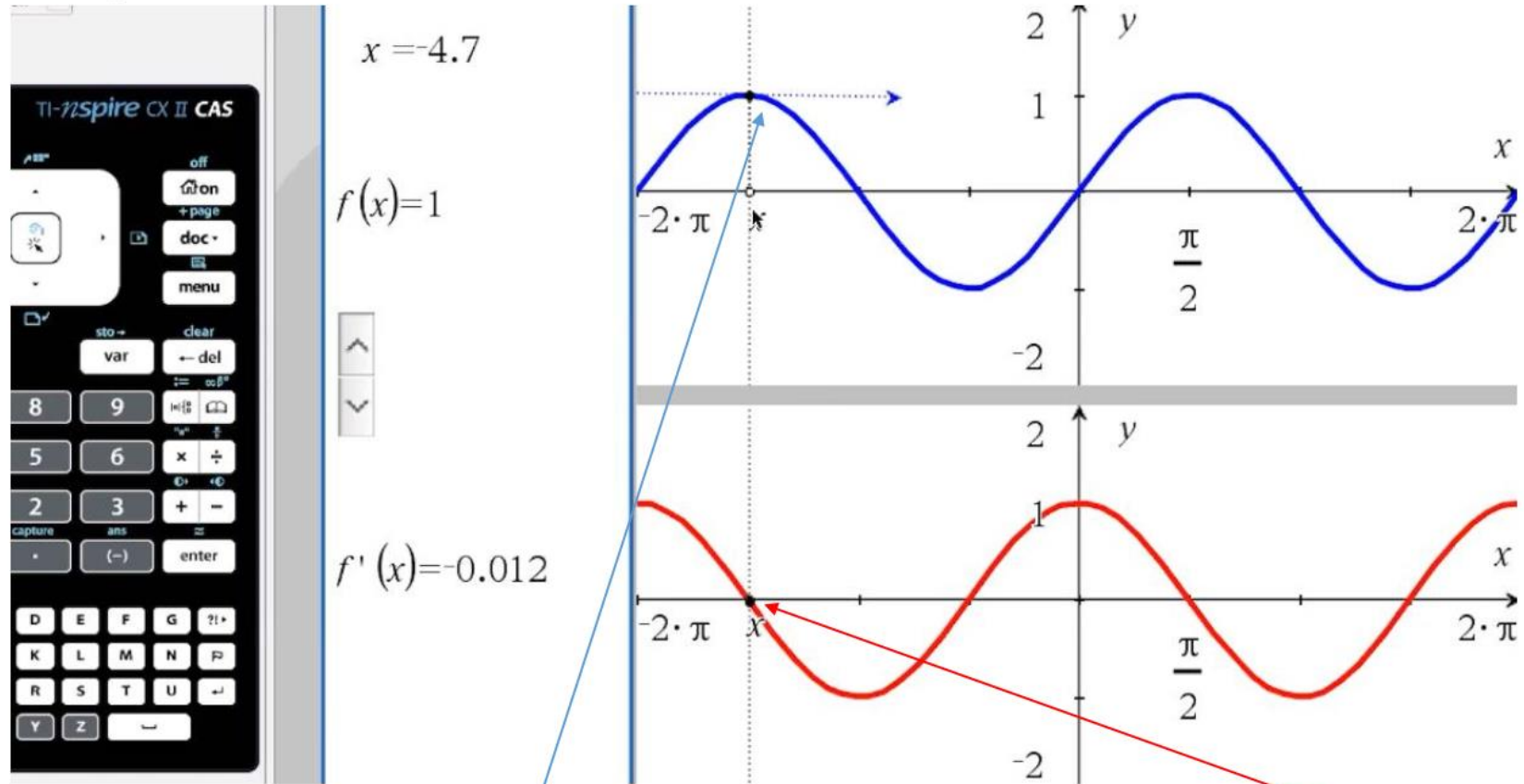
c) $h(x) = 7e^x + \sin x$



d) $j(x) = 3 \ln x - \cos x$

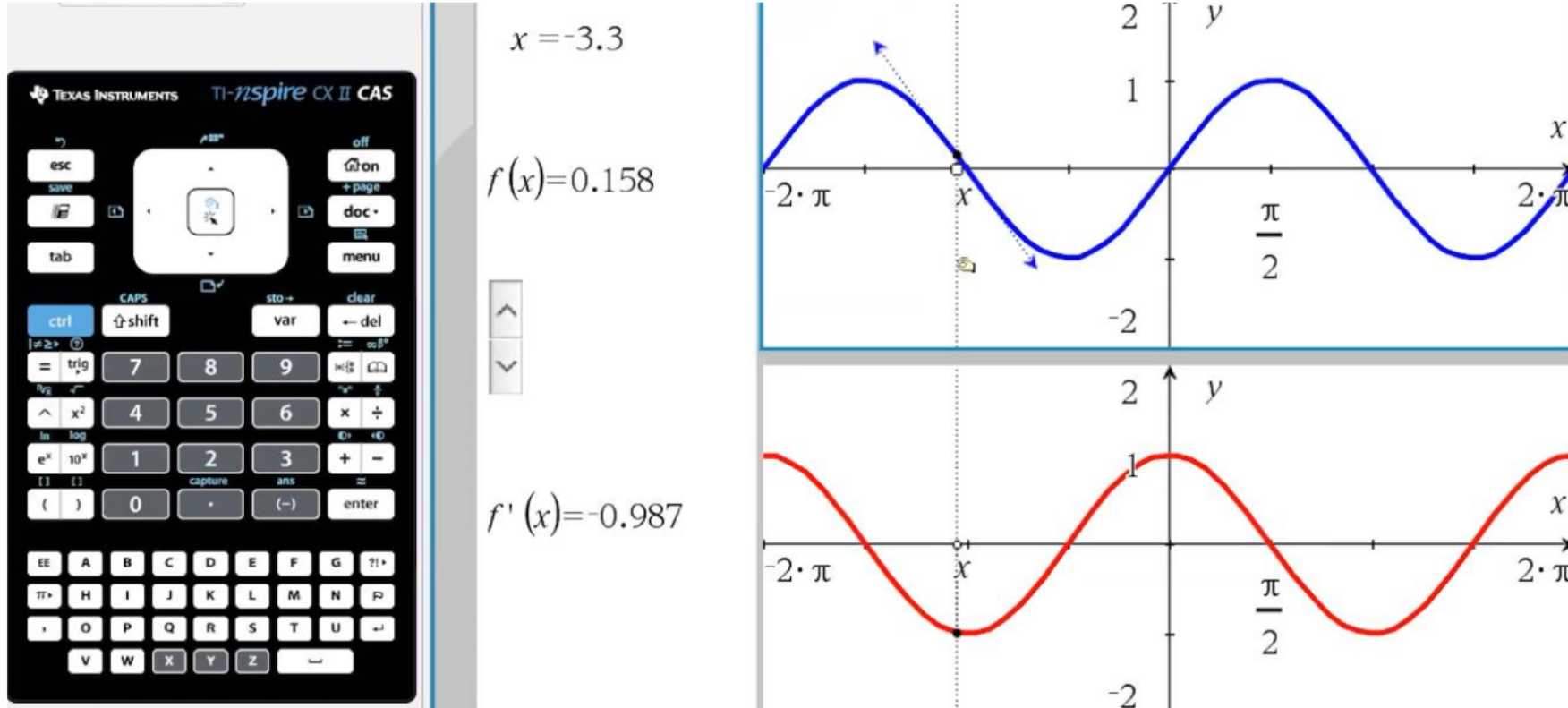


Why the derivative of $\sin x = \cos x$



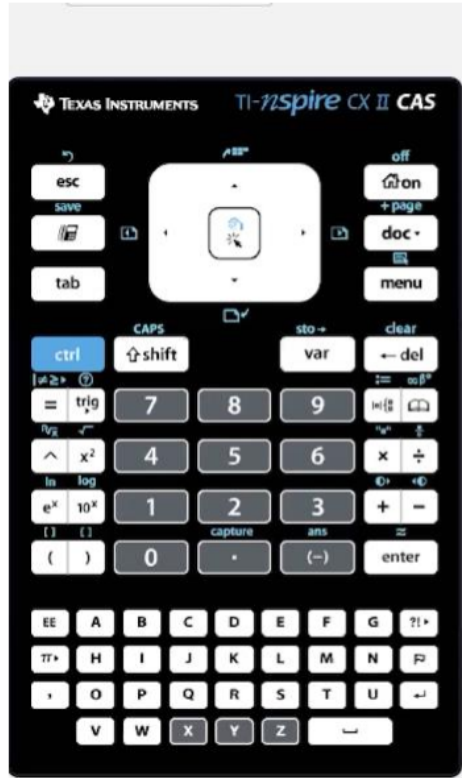
- Looking at the tangent drawn to $\sin x$ (blue), at the current position, the slope of the tangent line is 0.
- For $\cos x$ (in red) at the same x – *position*, we have a y-value of 0 because the value of the derivative is 0.

Why the derivative of $\sin x = \cos x$



- As we slide, the slope of the tangent line is negative, so that indicates that the value of the derivative is negative.
- So the graph of the derivative should be negative (part of the graph below the x-axis)

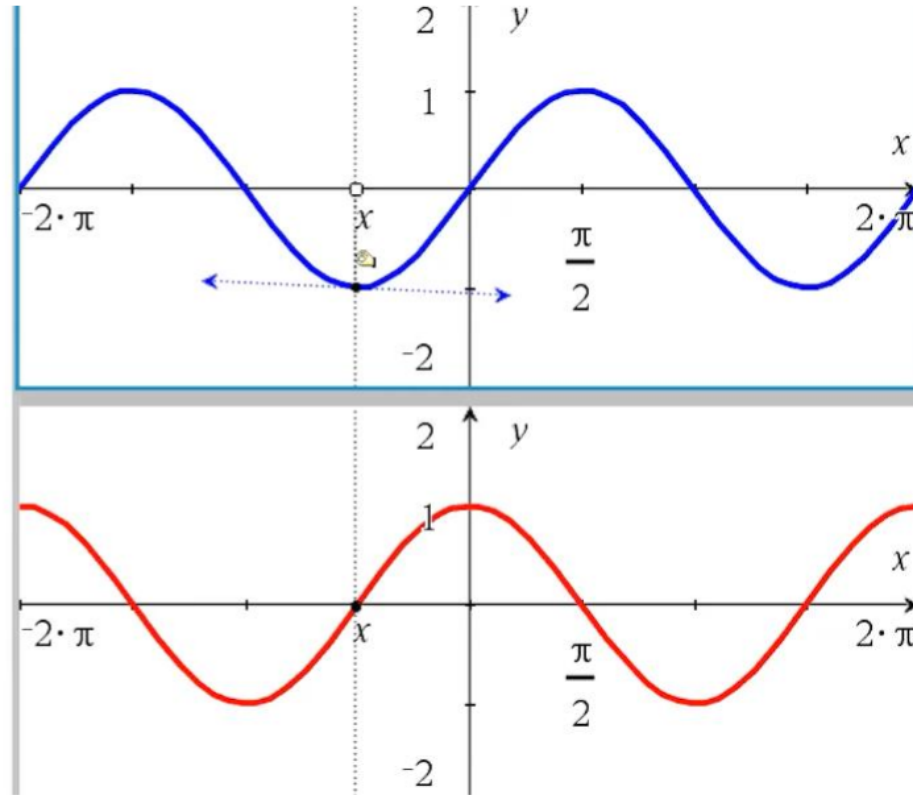
Why the derivative of $\sin x = \cos x$



$$x = -1.6$$

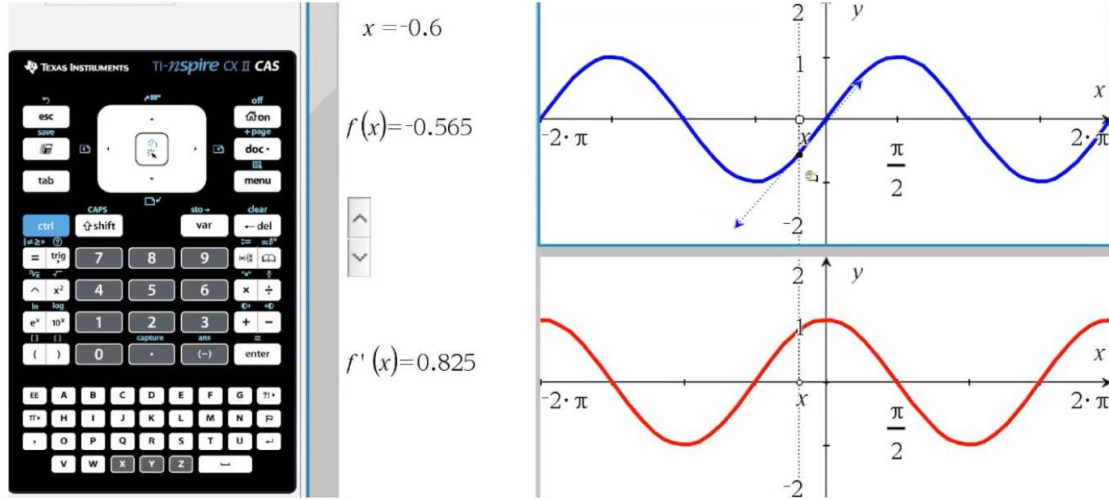
$$f(x) = -1$$

$$f'(x) = -0.029$$



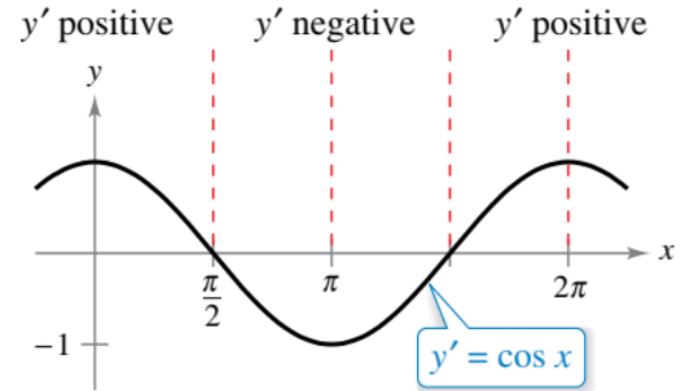
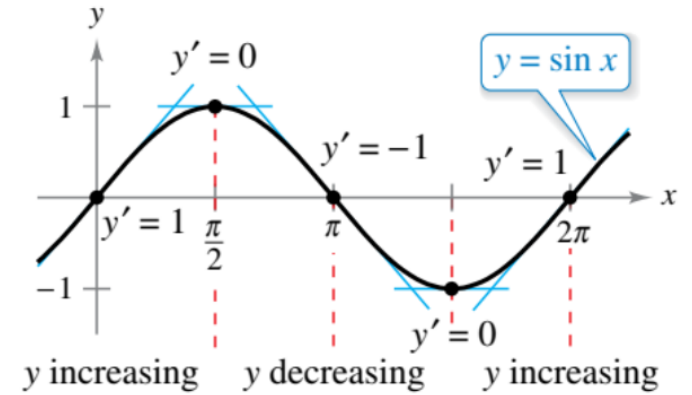
- As we slide, when we get to the point where it has a horizontal tangent line, we get back to the y-value of 0 on the cosine curve.

Why the derivative of $\sin x = \cos x$



➤ As we slide, when we get to a positive slope, it means the graph is going to lie above the x-axis until we get to another horizontal tangent.

Why the derivative of $\sin x = \cos x$



The derivative of the sine function is the cosine function.

Key Takeaways

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[e^x] = e^x$$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

These four functions are the four most common transcendental functions that you will encounter in calculus!