Turn and Talk

Let $f(x) = x^2$ with domain $x \geq 0$.

a) Find the range of f.

b) Find the domain and range of f^{-1}.

Let $f(x) = \sqrt{x-4}$.

a) Find the domain and range of f.

b) Find the domain and range of f^{-1}.

Let $f(x) = \dfrac{1}{x-2}$.

a) Find the domain and range of f.

b) State the domain and range of f^{-1}.

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b) State the domain and range of f^{-1}.

$f(x) = x^2, x \geq 0$

Domain of f: $[0, \infty)$

Range of f: $[0, \infty)$

Domain of f^{-1}: $[0, \infty)$

Range of f^{-1}: $[0, \infty)$

$f(x) = \sqrt{x-4}$

Domain of f: $[4, \infty)$

Range of f: $[0, \infty)$

Domain of f^{-1}: $[0, \infty)$

Range of f^{-1}: $[4, \infty)$

$f(x) = \dfrac{1}{x-2}$

Domain of f: $(-\infty, 2) \cup (2, \infty)$

Range of f: $(-\infty, 0) \cup (0, \infty)$

Domain of f^{-1}: $(-\infty, 0) \cup (0, \infty)$

Range of f^{-1}: $(-\infty, 2) \cup (2, \infty)$

Exponential Functions

- exponential function — 指数函数
- base — 底
- exponent — 指数
- growth — 增长
- decay — 衰减
- exponential growth — 指数增长
- exponential decay — 指数衰减
- initial value — 初始值
- growth factor — 增长因子
- decay factor — 衰减因子
- continuous growth — 连续增长

Exponential Functions

For example, if a population starts with P_0 individuals and then grows at an annual rate of 2%, its population after 1 year is

$$P(1) = \boxed{}$$

Its population after 2 years is

$$P(2) = \boxed{}.$$

In general, its population after t years is

$$P(t) = \boxed{}$$

, we graph both $y = x^2$ and $y = 2^x$ to show how the graphs differ.

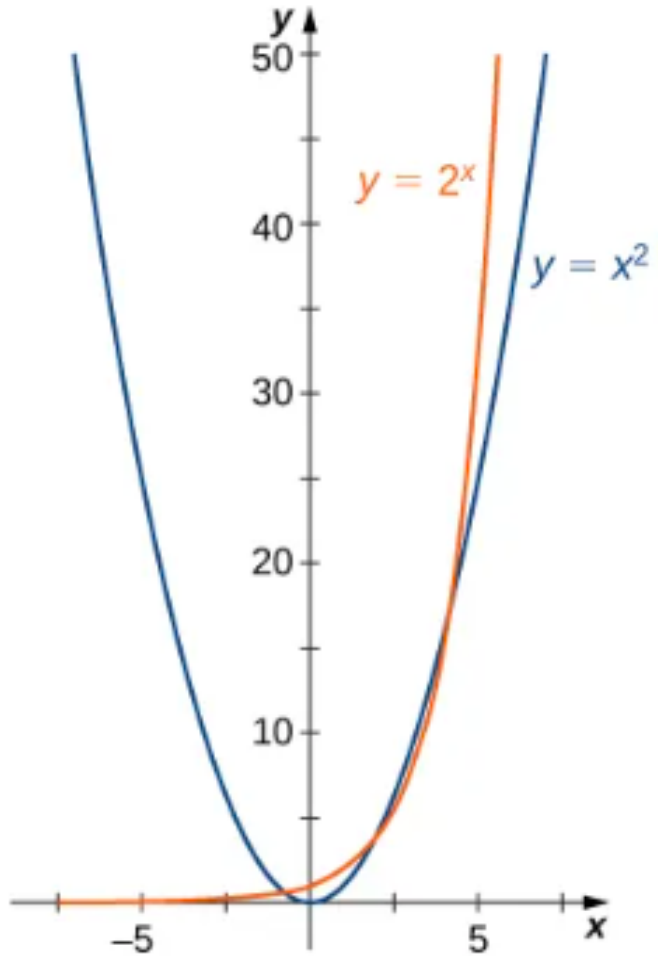


Figure 1.43 Both 2^x and x^2 approach infinity as $x \to \infty$, but 2^x grows more rapidly than x^2. As $x \to -\infty$, $x^2 \to \infty$, whereas $2^x \to 0$.

Bacterial Growth

Suppose a particular population of bacteria is known to double in size every 4 hours. If a culture starts with 1000 bacteria, the number of bacteria after 4 hours is $n(4) = 1000 \cdot 2$. The number of bacteria after 8 hours is $n(8) = n(4) \cdot 2 = 1000 \cdot 2^2$. In general, the number of bacteria after $4m$ hours is $n(4m) = 1000 \cdot 2^m$. Letting $t = 4m$, we see that the number of bacteria after t hours is $n(t) = 1000 \cdot 2^{t/4}$. Find the number of bacteria after 6 hours, 10 hours, and 24 hours.

Bacterial Growth

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The number of bacteria after 6 hours is given by $n\,(6) = 1000 \cdot 2^{6/4} \approx 2828$ bacteria. The number of bacteria after 10 hours is given by $n\,(10) = 1000 \cdot 2^{10/4} \approx 5657$ bacteria. The number of bacteria after 24 hours is given by $n\,(24) = 1000 \cdot 2^6 = 64{,}000$ bacteria.

Graphing Exponential Functions

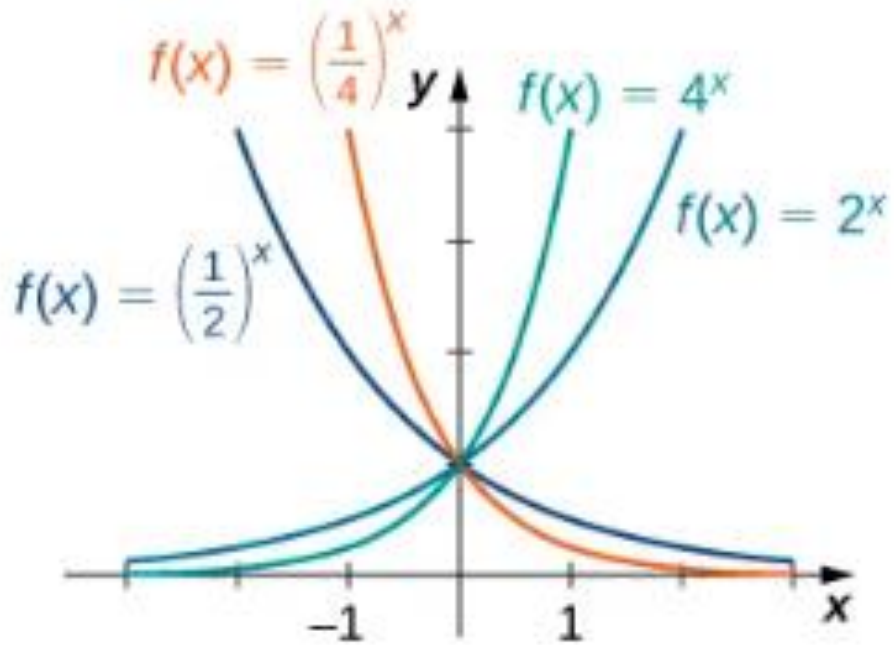
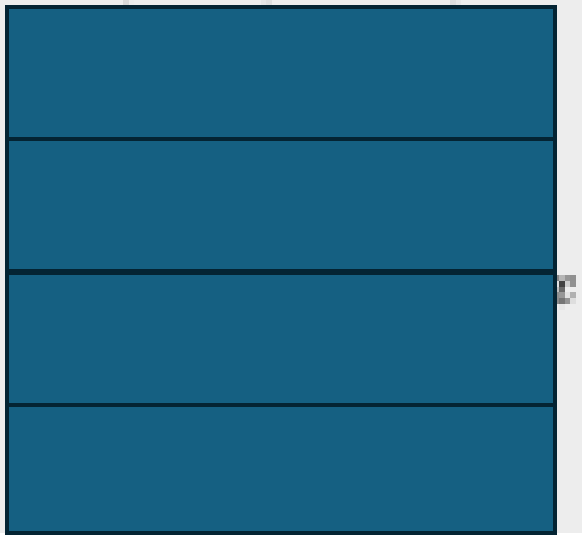


Figure 1.44 If $b > 1$, then b^x is increasing on $(-\infty, \infty)$. If $0 < b < 1$, then b^x is decreasing on $(-\infty, \infty)$.

RULE: LAWS OF EXPONENTS

For any constants $a > 0, b > 0$, and for all x and y,

1. $b^x \cdot b^y = b^{x+y}$



$$\left(\frac{a}{b}\right)^x = \underbrace{\frac{a}{b} \cdot \frac{a}{b} \cdot \ldots \cdot \frac{a}{b}}_{x \text{ times}}$$

Using the Laws of Exponents

Use the laws of exponents to simplify each of the following expressions.

a. $\dfrac{\left(2x^{2/3}\right)^3}{\left(4x^{-1/3}\right)^2}$

b. $\dfrac{\left(x^3y^{-1}\right)^2}{\left(xy^2\right)^{-2}}$

Using the Laws of Exponents

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b. $\dfrac{\left(x^3y^{-1}\right)^2}{\left(xy^2\right)^{-2}}$

a. We can simplify as follows:

$$\frac{\left(2x^{2/3}\right)^3}{\left(4x^{-1/3}\right)^2} = \frac{2^3\left(x^{2/3}\right)^3}{4^2\left(x^{-1/3}\right)^2} = \frac{8x^2}{16x^{-2/3}} = \frac{x^2x^{2/3}}{2} = \frac{x^{8/3}}{2}.$$

b. We can simplify as follows:

$$\frac{\left(x^3y^{-1}\right)^2}{\left(xy^2\right)^{-2}} = \frac{\left(x^3\right)^2\left(y^{-1}\right)^2}{x^{-2}\left(y^2\right)^{-2}} = \frac{x^6y^{-2}}{x^{-2}y^{-4}} = x^6x^2y^{-2}y^4 = x^8y^2.$$

The number **e**

money after 1 year is



The amount of money after 2 years is



More generally, the amount after t years is



If the money is compounded 2 times per year, the amount of money after half a year is



The amount of money after 1 year is



After t years, the amount of money in the account is



More generally, if the money is compounded n times per year, the amount of money in the account after t years is given by the function



What happens as $n \to \infty$? To answer this question, we let $m = n/r$ and write



m	10	100	1000	10,000	100,000	1,000,000
$\left(1+\frac{1}{m}\right)^m$						

Table 1.12 Values of $\left(1+\frac{1}{m}\right)^m$ as $m \to \infty$

Looking at this table, it appears that $(1+1/m)^m$ is approaching a number between 2.7 and 2.8 as $m \to \infty$. In fact, $(1+1/m)^m$ does approach some number as $m \to \infty$. We call this **number** e. To six decimal places of accuracy,

$$e \approx 2.718282.$$

Compounding Interest

Suppose $\$500$ is invested in an account at an annual interest rate of $r = 5.5\,\%$, compounded continuously.

a. Let t denote the number of years after the initial investment and $A(t)$ denote the amount of money in the account at time t. Find a formula for $A(t)$.
b. Find the amount of money in the account after 10 years and after 20 years.

Compounding Interest

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b. Find the amount of money in the account after 10 years and after 20 years.

a. If P dollars are invested in an account at an annual interest rate r, compounded continuously, then $A(t) = Pe^{rt}$. Here $P = \$500$ and $r = 0.055$. Therefore, $A(t) = 500e^{0.055t}$.

b. After 10 years, the amount of money in the account is

$$A(10) = 500e^{0.055 \cdot 10} = 500e^{0.55} \approx \$866.63.$$

After 20 years, the amount of money in the account is

$$A(20) = 500e^{0.055 \cdot 20} = 500e^{1.1} \approx \$1,502.08.$$

Logarithmic Functions

$$\log_b(x) = y \text{ if and only if } b^y = x.$$

Logarithmic Functions

- logarithmic function — 对数函数
- logarithm — 对数
- base — 底
- common logarithm — 常用对数
- natural logarithm — 自然对数
- inverse function — 反函数
- log rules (properties) — 对数性质
- change of base — 换底公式
- domain — 定义域
- asymptote — 渐近线

Since the functions $f(x) = e^x$ and $g(x) = \ln(x)$ are inverses of each other,

$$\ln(e^x) = x \text{ and } e^{\ln x} = x,$$

and their graphs are symmetric about the line $y = x$ (Figure 1.46).

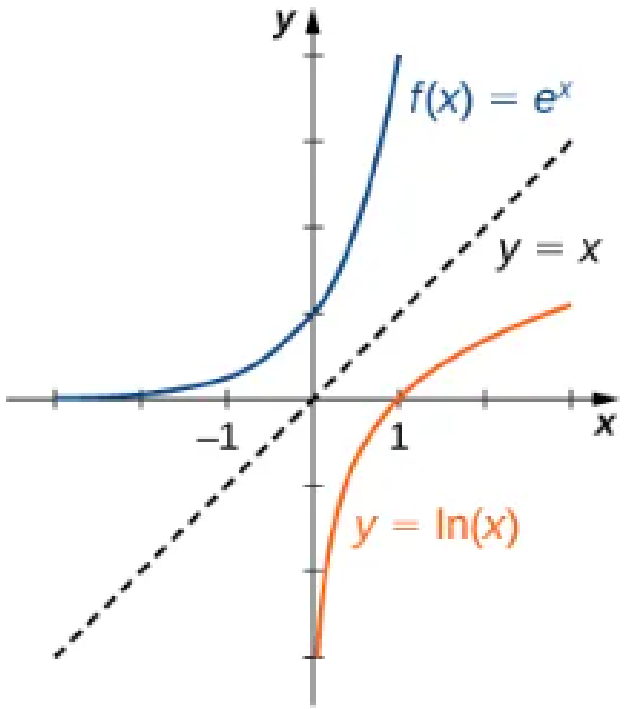
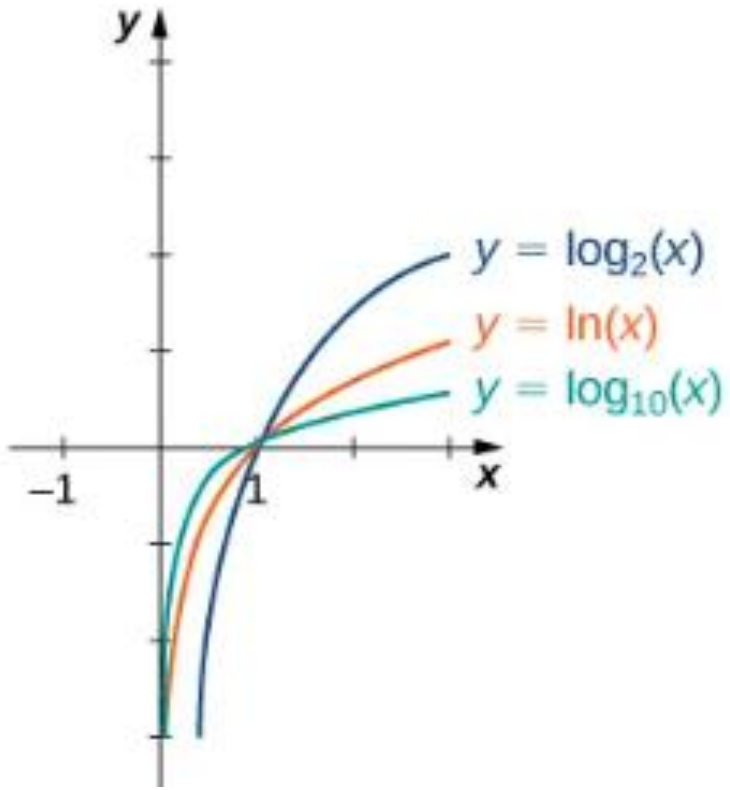


Figure 1.46 The functions $y = e^x$ and $y = \ln(x)$ are inverses of each other, so their graphs are symmetric about the line $y = x$.

In general, for any base $b > 0, b \neq 1$, the function $g(x) = \log_b(x)$ is symmetric about the line $y = x$ with the function $f(x) = b^x$. Using this fact and the graphs of the exponential functions, we graph functions $\log_b$ for several values of $b > 1$ (Figure 1.47).



RULE: PROPERTIES OF LOGARITHMS

If $a, b, c > 0, b \neq 1$, and r is any real number, then

1.	$\log_b(ac) = \log_b(a) + \log_b(c)$	(Product property)
2.	$\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$	(Quotient property)
3.	$\log_b(a^r) = r\log_b(a)$	(Power property)

Solving Equations Involving Exponential Functions

Solve each of the following equations for x.

a. $5^x = 2$

b. $e^x + 6e^{-x} = 5$

Solving Equations Involving Exponential Functions

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a. $5^x = 2$

b. $e^x + 6e^{-x} = 5$

Therefore, $x = \ln 2 / \ln 5$.

b. Multiplying both sides of the equation by e^x, we arrive at the equation

$$e^{2x} + 6 = 5e^x.$$

Rewriting this equation as

$$e^{2x} - 5e^x + 6 = 0,$$

we can then rewrite it as a quadratic equation in e^x:

$$(e^x)^2 - 5\,(e^x) + 6 = 0.$$

Now we can solve the quadratic equation. Factoring this equation, we obtain

$$(e^x - 3)\,(e^x - 2) = 0.$$

Therefore, the solutions satisfy $e^x = 3$ and $e^x = 2$. Taking the natural logarithm of both sides gives us the solutions $x = \ln 3, \ln 2$.

Solving Equations Involving Logarithmic Functions

Solve each of the following equations for x.

a. $\ln\left(\frac{1}{x}\right) = 4$

b. $\log_{10}\sqrt{x} + \log_{10}x = 2$

c. $\ln(2x) - 3\ln\left(x^2\right) = 0$

Solving Equations Involving Logarithmic Functions

Solve each of the following equations for x.

a. $\ln\left(\frac{1}{x}\right) = 4$

b. $\log_{10}\sqrt{x} + \log_{10}x = 2$

c. $\ln(2x) - 3\ln\left(x^2\right) = 0$

Therefore, the solution is $x = 1/e^4$.

b. Using the product and power properties of logarithmic functions, rewrite the left-hand side of the equation as

$$\log_{10}\sqrt{x} + \log_{10}x = \log_{10}x\sqrt{x} = \log_{10}x^{3/2} = \frac{3}{2}\log_{10}x.$$

Therefore, the equation can be rewritten as

$$\frac{3}{2}\log_{10}x = 2 \text{ or } \log_{10}x = \frac{4}{3}.$$

The solution is $x = 10^{4/3} = 10\sqrt[3]{10}$.

c. Using the power property of logarithmic functions, we can rewrite the equation as $\ln(2x) - \ln\left(x^6\right) = 0$.
Using the quotient property, this becomes

$$\ln\left(\frac{2}{x^5}\right) = 0.$$

Therefore, $2/x^5 = 1$, which implies $x = \sqrt[5]{2}$. We should then check for any extraneous solutions.

RULE: CHANGE-OF-BASE FORMULAS

Let $a > 0, b > 0$, and $a \neq 1, b \neq 1$.

1. $a^x = b^{x \log_b a}$ for any real number x.
   If $b = e$, this equation reduces to $a^x = e^{x \log_e a} = e^{x \ln a}$.
2. $\log_a x = \frac{\log_b x}{\log_b a}$ for any real number $x > 0$.
   If $b = e$, this equation reduces to $\log_a x = \frac{\ln x}{\ln a}$.

Use a calculating utility to evaluate $\log_3 7$ with the change-of-base formula presented earli

Use a calculating utility to evaluate $\log_3 7$ with the change-of-base formula presented earli

Use the second equation with $a = 3$ and $e = 3$:

$$\log_3 7 = \frac{\ln 7}{\ln 3} \approx 1.77124.$$

Hyperbolic Functions

Hyperbolic cosine

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic sine

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cosecant

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Hyperbolic secant

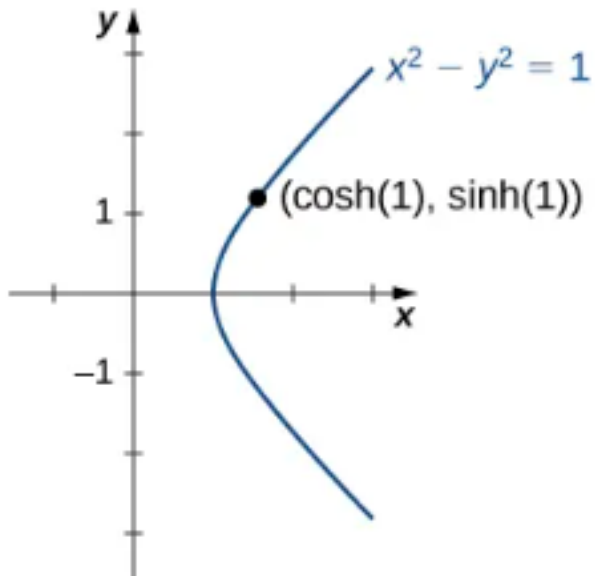
$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cotangent

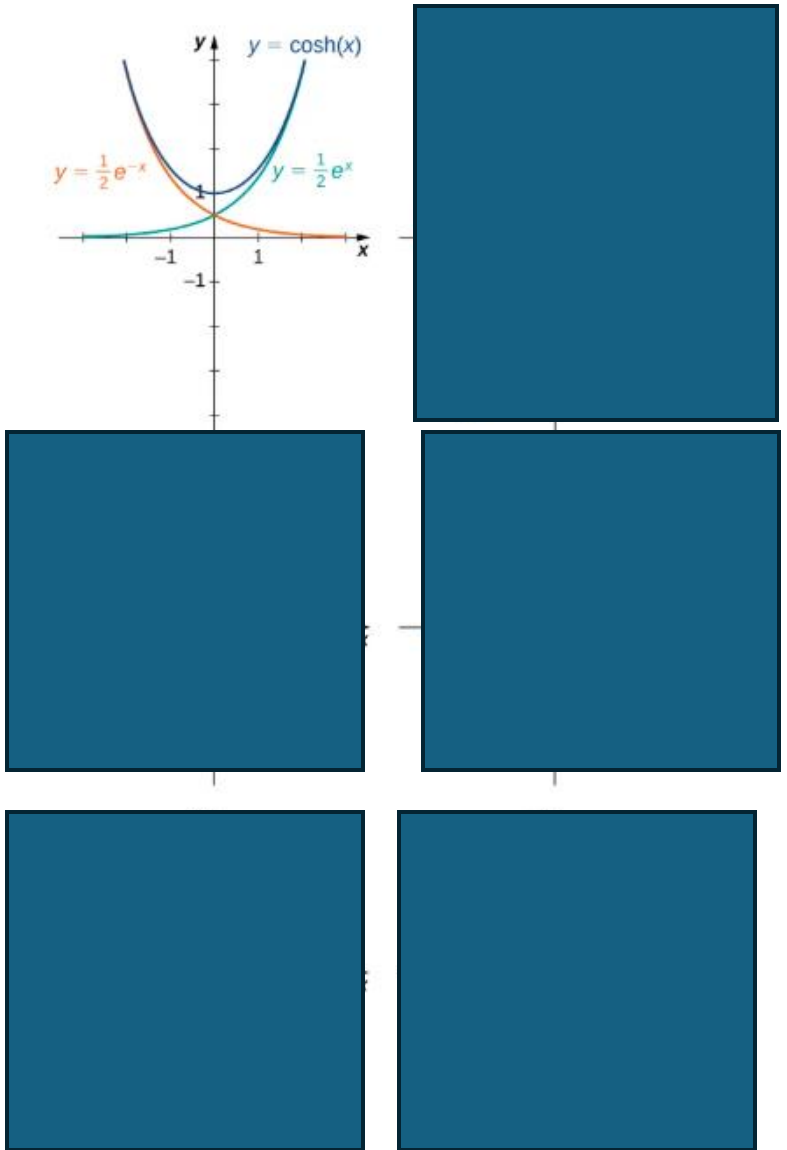
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\cosh^2 t - \sinh^2 t = \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} = 1.$$

This identity is the analog of the trigonometric identity $\cos^2 t + \sin^2 t = 1$. Here, given a value t, the point $(x, y) = (\cosh t, \sinh t)$ lies on the unit hyperbola $x^2 - y^2 = 1$ (Figure 1.50).



Graphs of Hyperbolic Functions



RULE: IDENTITIES INVOLVING HYPERBOLIC FUNCTIONS

1. $\cosh(-x) = \cosh x$
2. $\sinh(-x) = -\sinh x$
3. $\cosh x + \sinh x = e^x$
4. $\cosh x - \sinh x = e^{-x}$
5. $\cosh^2 x - \sinh^2 x = 1$
6. $1 - \tanh^2 x = \operatorname{sech}^2 x$
7. $\coth^2 x - 1 = \operatorname{csch}^2 x$
8. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
9. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

Evaluating Hyperbolic Functions

a. Simplify $\sinh(5\ln x)$.

b. If $\sinh x = 3/4$, find the values of the remaining five hyperbolic functions.

a. Using the definition of the sinh function, we write

$$\sinh(5\ln x) = \frac{e^{5\ln x} - e^{-5\ln x}}{2} = \frac{e^{\ln(x^5)} - e^{\ln(x^{-5})}}{2} = \frac{x^5 - x^{-5}}{2}.$$

b. Using the identity $\cosh^2 x - \sinh^2 x = 1$, we see that

$$\cosh^2 x = 1 + \left(\frac{3}{4}\right)^2 = \frac{25}{16}.$$

Since $\cosh x \geq 1$ for all x, we must have $\cosh x = 5/4$. Then, using the definitions for the other hyperbolic functions, we conclude that $\tanh x = 3/5, \operatorname{csch} x = 4/3, \operatorname{sech} x = 4/5,$ and $\coth x = 5/3$.

Inverse Hyperbolic Functions

DEFINITION

Inverse Hyperbolic Functions

$$\sinh^{-1}x = \operatorname{arcsinh}x = \ln\left(x + \sqrt{x^2+1}\right) \qquad \cosh^{-1}x = \operatorname{arccosh}x = \ln\left(x + \sqrt{x^2-1}\right)$$

$$\tanh^{-1}x = \operatorname{arctanh}x = \tfrac{1}{2}\ln\left(\tfrac{1+x}{1-x}\right) \qquad \coth^{-1}x = \operatorname{arccoth}x = \tfrac{1}{2}\ln\left(\tfrac{x+1}{x-1}\right)$$

$$\operatorname{sech}^{-1}x = \operatorname{arcsech}x = \ln\left(\tfrac{1+\sqrt{1-x^2}}{x}\right) \qquad \operatorname{csch}^{-1}x = \operatorname{arccsch}x = \ln\left(\tfrac{1}{x} + \tfrac{\sqrt{1+x^2}}{|x|}\right)$$

Evaluating Inverse Hyperbolic Functions

Evaluate each of the following expressions.

$$\sinh^{-1}(2)$$

$$\tanh^{-1}(1/4)$$

Evaluating Inverse Hyperbolic Functions

Evaluate each of the following expressions.

$$\sinh^{-1}(2)$$

$$\tanh^{-1}(1/4)$$

$$\sinh^{-1}(2) = \ln\left(2+\sqrt{2^2+1}\right) = \ln\left(2+\sqrt{5}\right) \approx 1.4436$$

$$\tanh^{-1}(1/4) = \tfrac{1}{2}\ln\left(\tfrac{1+1/4}{1-1/4}\right) = \tfrac{1}{2}\ln\left(\tfrac{5/4}{3/4}\right) = \tfrac{1}{2}\ln\left(\tfrac{5}{3}\right) \approx 0.2554$$