PRECALCULUS · FUNCTIONS

Combinations of Functions

Sums, products, quotients, composites — and inverses

10
questions worked

3
sections

8
true/false checks

Built from the 1-08 Combinations of Functions and 1-09 Inverse Functions booklets

How this lesson runs

Key concept, worked example, questions to try — then true or false

Starter	Two quick recall questions on the shape of quadratic functions — attempt before the reveal.
Key points	Each section opens with a summary of the ideas you need, main point highlighted first.
Question	Questions appear on their own slides. Attempt each before turning the page.
Solution	The next slide gives the condensed solution: the answer and the key steps only.
True or false	The lesson ends with eight statements — one per slide, answer on the slide that follows.

Key topic · Combinations of functions

What this lesson is about, in one page

The skill	Build new functions from old ones: add, subtract, multiply or divide two functions to make (f+g)(x), (f–g)(x), (fg)(x) or (f/g)(x).
The chain	Compose two functions: (f∘g)(x) = f(g(x)). The inner function runs first, then the outer function acts on the result.
The two domains	A combination lives only where BOTH functions are defined; a quotient must also exclude the zeros of the denominator.
The undo	An inverse function f^{-1} reverses f exactly: $f(f^{-1}(x)) = x$. Only one-to-one functions have inverses.

Vocabulary

Key terms with Chinese translations

combination	**组合**	a function built from two functions by +, −, × or ÷
sum (f+g)	**和**	$(f+g)(x) = f(x) + g(x)$
difference (f−g)	**差**	$(f-g)(x) = f(x) - g(x)$
product (fg)	**积**	$(fg)(x) = f(x)\cdot g(x)$
quotient (f/g)	**商**	$(f/g)(x) = f(x)/g(x)$, $g(x) \neq 0$
composition	**复合**	$(f\circ g)(x) = f(g(x))$, g acts first
inner function	**内函数**	the function applied first, g in f(g(x))
outer function	**外函数**	the function applied last, f in f(g(x))
one-to-one function	**一一对应函数**	each y comes from exactly one x
inverse function	**反函数**	f^{-1} reverses f: $f(f^{-1}(x)) = x$

S

Starter · The shape of quadratic functions

What you must remember before combining functions

- Two quick recall questions
- Vertex, axis of symmetry, direction of opening and the role of a

Key points · the shape of a quadratic

Everything about $y = ax^2 + bx + c$ in five lines

1	**The parabola**	Every quadratic graph is a parabola. $a > 0$ opens upward (a smile); $a < 0$ opens downward (a frown).
2	**Axis of symmetry**	The mirror line $x = -b/(2a)$. The vertex sits on it.
3	**Vertex**	At $x = -b/(2a)$. It is the minimum when $a > 0$ and the maximum when $a < 0$.
4	**Vertex form**	$y = a(x - h)^2 + k$ has vertex (h, k) — read the shifts straight off the brackets.
5	**y-intercept**	The curve crosses the y-axis at $(0, c)$ in standard form.

Question S1

5 marks

Starter · Shape of quadratics

THE QUESTION

For the quadratic $f(x) = 2x^2 - 8x + 6$:

(a) Does the parabola open upward or downward? How do you know?

(b) Find the equation of the axis of symmetry.

(c) Find the coordinates of the vertex.

(d) Find the y-intercept.

(e) Solve $f(x) = 0$ and state the roots.

Question S1 — solution

5 marks

Starter · Shape of quadratics

CONDENSED SOLUTION

- (a) Upward — $a = 2 > 0$.
- (b) $x = -b/(2a) = 8/4 = 2$.
- (c) Vertex $(2, f(2)) = (2, -2)$.
- (d) y-intercept $(0, 6)$.
- (e) $2(x-1)(x-3) = 0$, so roots $x = 1$ and $x = 3$.

TRAP Signs in $-b/(2a)$: with $b = -8$ the minus signs cancel, giving $+8$.

Question S2

4 marks

Starter · Shape of quadratics

THE QUESTION

The parabola $y = -x^2 + 4x - 3$ is drawn on the axes.

(a) State the direction of opening and the coordinates of the vertex.
(b) State the equation of the axis of symmetry.
(c) Factorise and write down the roots.
(d) Sketch the shape, labelling the vertex, both roots and the y-intercept.

Question S2 — solution

4 marks

Starter · Shape of quadratics

CONDENSED SOLUTION

- (a) $a = -1 < 0$, so it opens downward and the vertex is a maximum.
- Axis: $x = -4/(2\cdot(-1)) = 2$. Vertex: $y = -4 + 8 - 3 = 1$, so $(2, 1)$.
- (c) $-(x-1)(x-3) = 0$, so roots $x = 1$ and $x = 3$.
- (d) Downward parabola through $(0, -3)$, $(1, 0)$, $(2, 1)$, $(3, 0)$.
- Why this matters today: $x^2 - 4$ factors as $(x+2)(x-2)$ — and that factoring decides domains when we divide functions.

TRAP Read the sign of a from the coefficient in FRONT of x^2, not from the constant term.

01

Algebraic combinations

Sums, differences, products and quotients of functions

- $(f+g)(x) = f(x) + g(x)$, and the same pattern for $-$, $\times$ and $\div$
- The domain of a combination is the intersection of the two domains

Key points · algebraic combinations

The four operations, and the one rule about domains

```
(f+g)(x) = f(x) + g(x)      (f−g)(x) = f(x) − g(x)      (fg)(x) = f(x)·g(x)
(f/g)(x) = f(x)/g(x)
```

```
Domain of any combination  =  (domain of f) ∩ (domain of g)  −  and for a quotient also  g(x) ≠ 0
```

1. **Evaluate pointwise** — At every x, the new function does what f and g do, combined by the operation: (f+g)(3) = f(3) + g(3).

2. **Simplify the result** — (f+g)(x) for $f(x)=x^2-4$ and $g(x)=2x$ gives $x^2 + 2x - 4$. Collect like terms before stating the domain.

3. **Intersect the domains** — A combination is only defined where BOTH f and g are defined. Take the overlap of the two domains.

4. **Quotients lose more** — For (f/g)(x), every x with g(x) = 0 is removed as well — solve g(x) = 0 and exclude those values.

Worked example · algebraic combinations

the question

Question first — $f(x) = x^2 - 4$ and $g(x) = 2x$

THE QUESTION

Given $f(x) = x^2 - 4$ and $g(x) = 2x$, find the following and state the domain of each:

(a) $(f + g)(x)$ (b) $(fg)(x)$ (c) $(f/g)(x)$

Worked example · algebraic combinations — solution

the solution

Answers only — the method is the four definitions

CONDENSED SOLUTION

(a) $(f+g)(x) = x^2 + 2x - 4$, domain: all real x

(b) $(fg)(x) = 2x^3 - 8x$, domain: all real x

(c) $(f/g)(x) = (x^2 - 4)/(2x)$, domain: $x \neq 0$

All real x for (a) and (b) because $x^2 - 4$ and $2x$ are defined everywhere; the quotient drops $x = 0$ because that is where the denominator $2x$ is zero.

Try it yourself

your turn

Attempt before turning the page — answer on the next slide

YOUR TURN

Given $f(x) = x^2 - 1$ and $g(x) = 3x$:

Find $(f - g)(x)$, $(fg)(x)$ and $(f/g)(x)$,
and state the domain of $(f/g)(x)$.

Question Y1 — solution

3 marks

Try it · algebraic combinations

CONDENSED SOLUTION

- $(f-g)(x) = x^2 - 3x - 1$
- $(fg)(x) = 3x^3 - 3x$
- $(f/g)(x) = (x^2 - 1)/(3x)$, domain: $x \neq 0$

TRAP The denominator is 3x, so the excluded value is still x = 0 — the 3 does not change the zero of the denominator.

Question C1

3 marks

Algebraic combinations

THE QUESTION

Given $f(x) = x^2 + 3$ and $g(x) = 4x - 5$:

(a) Find $(f + g)(x)$ and simplify.
(b) Find $(fg)(x)$.
(c) State the domain of $(f + g)(x)$ and of $(fg)(x)$.

Question C1 — solution

3 marks

Algebraic combinations

CONDENSED SOLUTION

- (a) $x^2 + 4x - 2$
- (b) $4x^3 - 5x^2 + 12x - 15$
- (c) All real x for both — polynomials are defined everywhere.

TRAP Polynomials never restrict a domain. Restrictions come from denominators, square roots and logarithms — none appear here.

Question C2

4 marks

Algebraic combinations

THE QUESTION

Given $f(x) = \sqrt{x}$ and $g(x) = 1/x$:

(a) Find $(f + g)(x)$.
(b) Find the domain of f and of g.
(c) Hence state the domain of $(f + g)(x)$.
(d) Find $(f/g)(x)$ and state its domain.

Question C2 — solution

4 marks

Algebraic combinations

CONDENSED SOLUTION

- (a) $\sqrt{x} + 1/x$
- (b) f: $x \geq 0$; g: $x \neq 0$
- (c) Domain of (f+g): $x > 0$
- (d) $(f/g)(x) = x\sqrt{x}$, domain: $x > 0$

TRAP The intersection $x \geq 0 \cap x \neq 0$ is $x > 0$. Zero sneaks out of the domain even though neither formula “looks” undefined there.

02

Composition of functions

One function inside another: $(f \circ g)(x) = f(g(x))$

- The inner function acts first, then the outer one
- Order matters: $(f \circ g)(x)$ and $(g \circ f)(x)$ are usually different

Key points · composition of functions

The chain rule of function building

$(f \circ g)(x) = f(g(x))$ — read as “f of g of x”: g acts FIRST, then f

In general $(f \circ g)(x) \neq (g \circ f)(x)$ · domain of $f \circ g$: x must be in the domain of g AND g(x) in the domain of f

1. **Work inside out** — To find $(f \circ g)(2)$: compute g(2) first, then feed that number into f. The inner function always runs first.
2. **Substitute the whole rule** — To find an expression for $(f \circ g)(x)$, replace every x in f with the ENTIRE rule for g — brackets are essential.
3. **Order matters** — $f(x)=2|x|$ and $g(x)=x+3$ give $(f \circ g)(x)=2|x+3|$ but $(g \circ f)(x)=2|x|+3$. Composing is not commutative.
4. **Respect both domains** — x must be allowed into g, and g(x) must be allowed into f. For $f(x)=\sqrt{x}$ inside, we need $g(x) \geq 0$.

Worked example · composition

the question

Question first — $f(x) = 2|x|$ and $g(x) = x + 3$

THE QUESTION

Given $f(x) = 2|x|$ and $g(x) = x + 3$:

(a) Find $(f \circ g)(x)$.

(b) Find $(g \circ f)(x)$.

(c) Are $(f \circ g)(x)$ and $(g \circ f)(x)$ equal? Explain.

Worked example · composition — solution

the solution

Answers only — inside first, then outside

CONDENSED SOLUTION

(a) $(f \circ g)(x) = f(x+3) = 2|x + 3|$

(b) $(g \circ f)(x) = g(2|x|) = 2|x| + 3$

(c) No — composition is not commutative.

In (a) the +3 happens inside the absolute value; in (b) it happens after. Same ingredients, different order, different function — a favourite exam trap.

Try it yourself

your turn

Attempt before turning the page — answer on the next slide

YOUR TURN

Given $f(x) = \sqrt[3]{x}$ and $g(x) = (8 - x)/5$:

Find $(f \circ g)(x)$ and evaluate it at $x = 3$.

Question Y2 — solution

3 marks

Try it · composition

CONDENSED SOLUTION

- $(f \circ g)(x) = \sqrt[3]{((8 - x)/5)}$
- At x = 3: $\sqrt[3]{((8 - 3)/5)} = \sqrt[3]{1} = 1$

TRAP Substitute the WHOLE rule for g. A common slip is cubing only the 8 — brackets protect the entire expression.

Question C3

3 marks

Composition of functions

THE QUESTION

Given $f(x) = x^2$ and $g(x) = 3x - 1$:

(a) Find $(f \circ g)(x)$.
(b) Find $(g \circ f)(x)$.
(c) Evaluate $(f \circ g)(2)$.

Question C3 — solution

3 marks

Composition of functions

CONDENSED SOLUTION

- (a) $(f \circ g)(x) = (3x - 1)^2$
- (b) $(g \circ f)(x) = 3x^2 - 1$
- (c) $(f \circ g)(2) = (6 - 1)^2 = 25$

TRAP In (a) the whole expression $3x - 1$ is squared; in (b) only x^2 is multiplied by 3. Different functions, same two ingredients.

Question C4

4 marks

Composition of functions

THE QUESTION

Given $f(x) = \sqrt{x}$ and $g(x) = 1/x$:

(a) Find $(f \circ g)(x)$.
(b) State the domain of g.
(c) Explain why the domain of $(f \circ g)(x)$ is $x > 0$, not $x \neq 0$.
(d) Evaluate $(f \circ g)(4)$.

Question C4 — solution

4 marks

Composition of functions

CONDENSED SOLUTION

- (a) $(f \circ g)(x) = \sqrt{(1/x)}$
- (b) Domain of g: $x \neq 0$
- (c) Also need $g(x) > 0$ for $\sqrt{}$: $1/x > 0$ means $x > 0$
- (d) $(f \circ g)(4) = \sqrt{(1/4)} = 1/2$

TRAP Both gates apply: x must enter g ($x \neq 0$) AND g(x) must enter f ($\sqrt{}$ needs $1/x \geq 0$). Together: $x > 0$.

03

Inverse functions

The function that undoes another function

- Only one-to-one functions have inverses — the horizontal line test
- $f(f^{-1}(x)) = f^{-1}(f(x)) = x$, and the graph reflects over $y = x$

Key points · inverse functions

Four ideas that decide whether an inverse exists

1	**One-to-one**	Each y comes from exactly one x. Graph test: no horizontal line crosses the curve more than once.
2	**The undo property**	$f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ for every x in the right domain. Composing a function with its inverse gives the identity.
3	**Domain and range swap**	Domain of f^{-1} = range of f; range of f^{-1} = domain of f. Restrictions follow the swap.
4	**Mirror over $y = x$**	The graph of f^{-1} is the graph of f reflected over the line $y = x$. Every point (a, b) becomes (b, a).

Formulae · finding an inverse function

The three-step recipe you will use in every inverse question

Step 1 write $y = f(x)$ **Step 2** swap x and y **Step 3** solve for y → $f^{-1}(x)$

Check: $f(f^{-1}(x)) = x$ · $f^{-1}(f(x)) = x$ — if either fails, the inverse is wrong

1 **Example: f(x) = 2x + 3**

$y = 2x + 3 \rightarrow x = 2y + 3 \rightarrow y = (x - 3)/2$, so $f^{-1}(x) = (x - 3)/2$.

2 **Why swap x and y?**

The inverse sends outputs back to inputs: the old y becomes the new x. Swapping is the algebraic mirror in $y = x$.

3 **Powers and roots**

$f(x) = x^3$ inverts to $f^{-1}(x) = \sqrt[3]{x}$. In general, the nth power swaps with the nth root (with domain care for even n).

4 **Negative example**

$f(x) = -\sqrt{(x + 2)}$ has range $y \le 0$, so its inverse has domain $x \le 0$ — the range of f becomes the domain of f^{-1}.

Worked example · inverse functions

the question

Question first — f(x) = 2x + 3

THE QUESTION

Given $f(x) = 2x + 3$:

(a) Find $f^{-1}(x)$.

(b) Verify that f and f^{-1} are inverses by showing $f(f^{-1}(x)) = x$.

Worked example · inverse functions — solution

the solution

Answers only — swap, solve, verify

CONDENSED SOLUTION

(a) $f^{-1}(x) = (x - 3)/2$

(b) $f(f^{-1}(x)) = 2\cdot((x - 3)/2) + 3 = (x - 3) + 3 = x$ ✓

Swap: $x = 2y + 3$. Solve: $2y = x - 3$, $y = (x - 3)/2$. The verification composes f with f^{-1} and must land exactly on x — write that final "$= x$", it is the mark.

Question C5

3 marks

Inverse functions

THE QUESTION

Given $f(x) = x^3 + 10$:

(a) Find $f^{-1}(x)$.

(b) Verify that $f(f^{-1}(x)) = x$.

(c) State the domain and range of f^{-1}.

Question C5 — solution

3 marks

Inverse functions

CONDENSED SOLUTION

- (a) $f^{-1}(x) = \sqrt[3]{(x - 10)}$
- (b) $f(\sqrt[3]{(x - 10)}) = (\sqrt[3]{(x - 10)})^3 + 10 = x - 10 + 10 = x$ ✓
- (c) Domain and range: all real x

TRAP The cube of a cube root is the inside — that is why the ± 10 cancels. Cubes and cube roots are inverses for ALL real numbers.

Question C6

4 marks

Inverse functions

THE QUESTION

Let $f(x) = -\sqrt{(x+2)}$, which has inverse $f^{-1}(x) = x^2 - 2$ valid for $x \leq 0$.

(a) State the domain and range of f.

(b) State the domain and range of f^{-1}.

(c) Verify that $f(f^{-1}(x)) = x$ for $x \leq 0$.

(d) How do the domains and ranges of f and f^{-1} relate?

Question C6 — solution

4 marks

Inverse functions

CONDENSED SOLUTION

- (a) f: domain $x \geq -2$, range $y \leq 0$
- (b) f^{-1}: domain $x \leq 0$, range $y \geq -2$
- (c) $f(x^2 - 2) = -\sqrt{(x^2 - 2 + 2)} = -\sqrt{(x^2)} = -|x| = x$ for $x \leq 0$ ✓
- (d) Swapped: domain of f^{-1} is the range of f.

TRAP The negative square root forces the range $y \leq 0$. After the swap, that range becomes the domain $x \leq 0$ of the inverse.

Question C7

3 marks

Inverse functions · verify a pair

THE QUESTION

Show that $f(x) = 4x + 7$ and $g(x) = (x - 7)/4$ are inverses of each other.

Verify BOTH compositions: f(g(x)) and g(f(x)).

Question C7 — solution

3 marks

Inverse functions · verify a pair

CONDENSED SOLUTION

- $f(g(x)) = 4\cdot(x - 7)/4 + 7 = x - 7 + 7 = x$ ✓
- $g(f(x)) = (4x + 7 - 7)/4 = 4x/4 = x$ ✓
- Both compositions give x, so f and g are inverses.

TRAP "Verify they are inverses" means BOTH orders. One composition on its own is only half the proof — the second mark hides in g(f(x)).

Mark scheme

Marks by question, with what earns each mark

S1	**5**	1 per part: direction, axis, vertex, intercept, roots
S2	**4**	1 direction+vertex, 1 axis, 1 roots, 1 labelled sketch
Y1	**3**	1 each: difference, product, quotient with domain $x \neq 0$
C1	**3**	1 sum simplified · 1 product · 1 domain statement (all real)
C2	**4**	1 sum · 1 both domains · 1 intersection $x > 0$ · 1 quotient + domain
Y2	**3**	1 composition rule · 1 substitution · 1 evaluation = 1
C3	**3**	1 each: $(f \circ g)$, $(g \circ f)$, evaluation 25
C4	**4**	1 rule $\sqrt{(1/x)}$ · 1 domain of g · 1 explanation $x > 0$ · 1 evaluation 1/2
C5	**3**	1 inverse $\sqrt[3]{(x - 10)}$ · 1 verification · 1 domain/range all real
C6	**4**	1 f domain/range · 1 f^{-1} domain/range · 1 verification · 1 swap statement
C7	**3**	1 $f(g(x)) = x$ · 1 $g(f(x)) = x$ · 1 conclusion

Main lesson total: 30 marks. Starter: 9 practice marks (not assessed).

Answer key

One line per question

ANSWERS

- S1 up; x = 2; (2, −2); (0, 6); roots 1 and 3
- S2 down; vertex (2, 1); x = 2; roots 1 and 3
- Y1 $x^2 - 3x - 1 \cdot 3x^3 - 3x \cdot (x^2-1)/(3x)$, $x \neq 0$
- C1 $x^2 + 4x - 2 \cdot 4x^3 - 5x^2 + 12x - 15 \cdot$ all real
- C2 $\sqrt{x} + 1/x \cdot f$: $x \geq 0$, g: $x \neq 0 \cdot x > 0 \cdot x\sqrt{x}$, $x > 0$
- Y2 $\sqrt[3]{((8 - x)/5)}$; at x = 3 gives 1

QUICK VERIFICATION

- C3 $(3x - 1)^2 \cdot 3x^2 - 1 \cdot 25$
- C4 $\sqrt{(1/x)} \cdot x \neq 0 \cdot x > 0 \cdot 1/2$
- C5 $\sqrt[3]{(x - 10)} \cdot$ verified $\cdot$ all real
- C6 $x \geq -2$, $y \leq 0 \cdot x \leq 0$, $y \geq -2 \cdot$ verified for $x \leq 0$
- C7 both compositions give x ✓
- Quick check: composing f with f^{-1}
- must return x in BOTH orders.

True or false · 1

T / F

Decide, then justify your decision — answer on the next slide

$(f \circ g)(x) = (g \circ f)(x)$ for all functions f and g.

True or false · 1 — answer

FALSE

FALSE

Composition is not commutative. With $f(x) = 2|x|$ and $g(x) = x + 3$: $(f \circ g)(x) = 2|x+3|$ but $(g \circ f)(x) = 2|x| + 3$ — different functions.

True or false · 2

T / F

Decide, then justify your decision — answer on the next slide

The domain of $f \circ g$ must respect the domain of g.

True or false · 2 — answer

TRUE

TRUE

x has to get INTO g first, so x must be in the domain of g; and g(x) must then be in the domain of f. Both gates apply.

True or false · 3

T / F

Decide, then justify your decision — answer on the next slide

$f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

True or false · 3 — answer

TRUE

TRUE

That is the defining property of an inverse: composing f with f^{-1} returns the input. The same holds for $f^{-1}(f(x)) = x$.

True or false · 4

T / F

Decide, then justify your decision — answer on the next slide

A function whose outputs repeat can still have an inverse.

True or false · 4 — answer

FALSE

FALSE

A repeated output means two x-values share one y — not one-to-one. The inverse would send that y back to two different places, which is not a function.

True or false · 5

T / F

Decide, then justify your decision — answer on the next slide

The graph of f^{-1} is the graph of f reflected over the line y = x.

True or false · 5 — answer

TRUE

TRUE

Every point (a, b) on f swaps to (b, a) on f^{-1}. Swapping coordinates is exactly a reflection over y = x.

True or false · 6

T / F

Decide, then justify your decision — answer on the next slide

The domain of (f/g)(x) excludes the zeros of the denominator.

True or false · 6 — answer

TRUE

TRUE

Division by zero is undefined, so every x with g(x) = 0 must be removed — on top of the intersection of the two domains.

True or false · 7

T / F

Decide, then justify your decision — answer on the next slide

The inverse function swaps the domain and the range of f.

True or false · 7 — answer

TRUE

TRUE

Domain of f^{-1} = range of f, and range of f^{-1} = domain of f. For $f(x) = -\sqrt{(x+2)}$: domain $x \geq -2$, range $y \leq 0$ — swapped in the inverse.

True or false · 8

T / F

Decide, then justify your decision — answer on the next slide

$f(x) = x^2$ has an inverse function for all real x.

True or false · 8 — answer

FALSE

FALSE

It fails the horizontal line test: $f(2) = f(-2) = 4$. Restrict the domain to $x \geq 0$ and then $f^{-1}(x) = \sqrt{x}$ exists.

FIVE THINGS TO REMEMBER

Do these and combinations of functions will not cost you marks

- The domain of any combination is the intersection of the two domains.
- For quotients, also exclude every zero of the denominator.
- Composition works inside first: $(f \circ g)(x) = f(g(x))$, and order matters.
- Find f^{-1} in three steps: write $y = f(x)$, swap x and y, solve for y.
- Verify inverses with BOTH compositions — each must return x.

Built from the 1-08 Combinations of Functions and 1-09 Inverse Functions booklets.