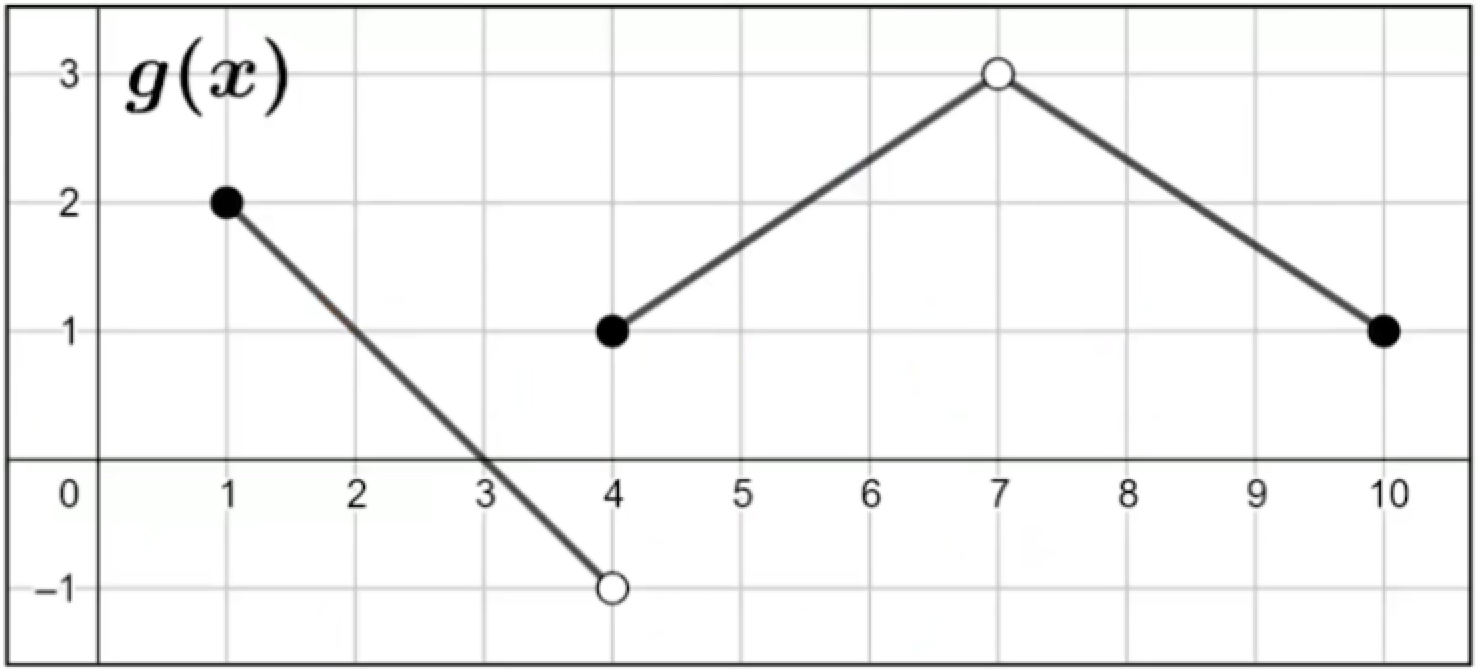
Left and Right Sided Limits



A) $\lim_{x \to 7^-} g(x) =$

B) $\lim_{x \to 7^+} g(x) =$

C) $\lim_{x \to 4^-} g(x) =$

D) $\lim_{x \to 4^+} g(x) =$

Determining Limits

1. **Limit — 极限**
2. **Approaches / Tends to — 趋近于**
3. **Left-hand limit — 左极限**
4. **Right-hand limit — 右极限**
5. **Indeterminate form — 不定式**

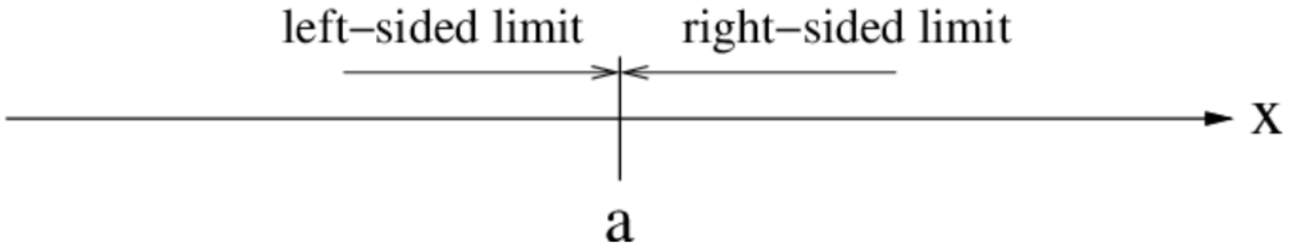
UNIT 1 KNOWLEDGE – CALCULUS 12 – LIMITS AND CONTINUITY

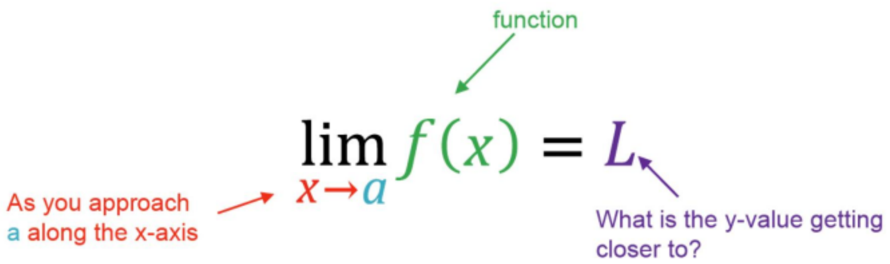
- 1.1 Introducing Calculus: Can Change Occur at an Instant? ✓
- 1.2 Defining Limits and Using Limit Notation ✓
- 1.3 Estimating Limit Values from Graphs ✓
- 1.4 Estimating Limit Values from Tables ✓
- 1.5 Determining Limits Using Algebraic Properties of Limits
- 1.6 Determining Limits Using Algebraic Manipulation
- 1.7 Selecting Procedures for Determining Limits
- 1.8 Determining Limits Using the Squeeze Theorem
- 1.9 Connecting Multiple Representations of Limits
- 1.10 Exploring Types of Discontinuities
- 1.11 Defining Continuity at a Point
- 1.12 Confirming Continuity over an Interval
- 1.13 Removing Discontinuities
- 1.14 Connecting Infinite Limits and Vertical Asymptotes
- 1.15 Connecting Limits at Infinity and Horizontal Asymptotes
- 1.16 Working with the Intermediate Value Theorem (IVT)

What Will We Learn?

- How can we determine the limits of functions using limit theorems?

- How can we evaluate the limits of piecewise functions using left and right-sided limits algebraically?





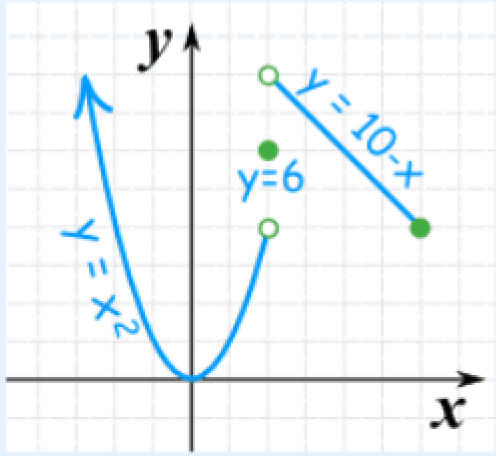
Piecewise function:

- A function with multiple pieces of curves in its graph. It means it has different definitions depending upon the value of the input.
- It can be a function made up of 3 pieces.

Example:

- when x is less than 2, it gives $\mathbf{x^2}$,
- when x is exactly 2 it gives **6**
- when x is more than 2 and less than or equal to 6 it gives the line **10-x**

It looks like this:



(a solid dot means "including",
an open dot means "not including")

And this is how we write it:

$$f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 6 & \text{if } x = 2 \\ 10 - x & \text{if } x > 2 \text{ and } x \le 6 \end{cases}$$

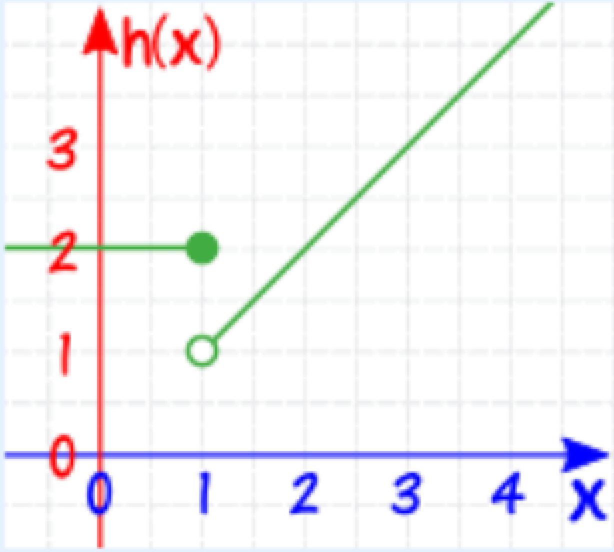
Piecewise function:

- A function with multiple pieces of curves in its graph. It means it has different definitions depending upon the value of the input.

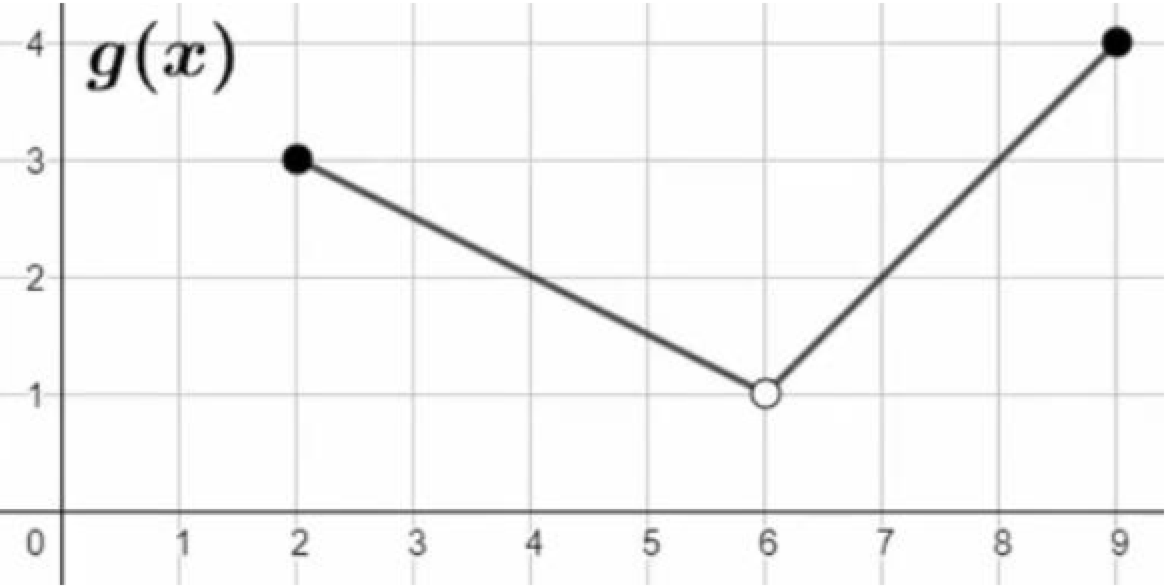
Example: Here is another piecewise function:

$$h(x) = \begin{cases} 2, & \text{if } x \leq 1 \\ x, & \text{if } x > 1 \end{cases}$$

which looks like:



Determining Limits Using Algebraic Properties of Limits

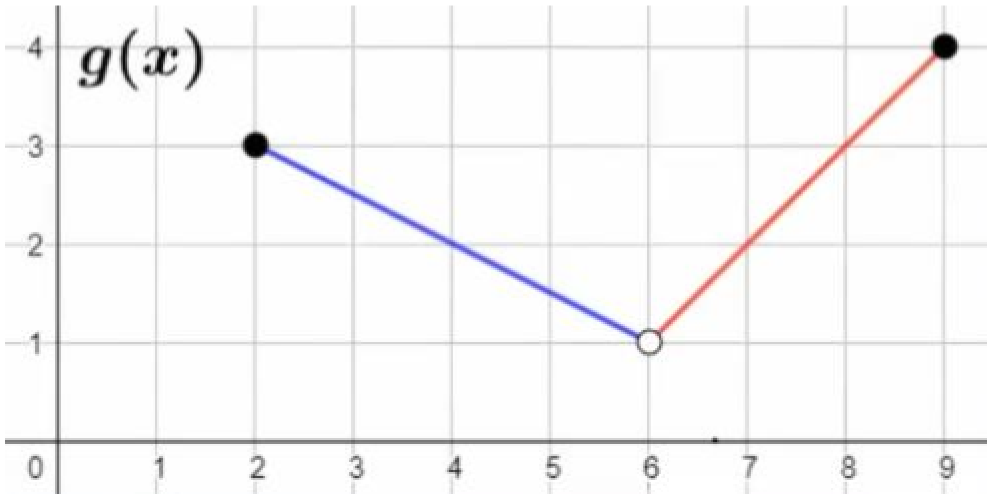


$$\lim_{x \to 6^-} g(x) = \square \qquad \lim_{x \to 6^+} g(x) = \square$$

- Notice there’s a hole on the graph
- Note ***g(6) is undefined***….***BUT this did not affect the value of the limit***

Determining Limits Using Algebraic Properties of Limits

Graphical Representation



$$\lim_{x \to 6^-} g(x) = 1$$

$$\lim_{x \to 6^+} g(x) = 1$$

Algebraic Representation (e.g. piecewise function)

$$g(x) = \begin{cases} -\frac{1}{2}x + 4, & x < 6 \\ x - 5, & x > 6 \end{cases}$$

left

right

- Notice that we can't include 6 because there is a hole on the graph. i.e. Never touches 6

Determining Limits Using Algebraic Properties of Limits

Algebraic Representation (e.g. piecewise function)

$$g(x) = \begin{cases} -\frac{1}{2}x + 4, & x < 6 \\ x - 5, & x > 6 \end{cases}$$

$$\lim_{x \to 6^-} g(x) = \lim_{x \to 6^-} \left(-\frac{1}{2}x + 4 \right) =$$

$$\lim_{x \to 6^+} g(x) = \lim_{x \to 6^+} (x - 5) =$$

- ***Even with no graph*** we can still evaluate the function by **plugging in the x value** it gets close to from the left & right and see if the answers are the same.

Determining Limits Using Algebraic Properties of Limits

Algebraic Representation (e.g. piecewise function)

$$g(x) = \begin{cases} -\frac{1}{2}x + 4, & x < 6 \\ x - 5, & x > 6 \end{cases}$$

$$\lim_{x \to 6^-} g(x) = \lim_{x \to 6^-} \left(-\frac{1}{2}x + 4\right) = -\frac{1}{2}(6) + 4 = 1$$

$$\lim_{x \to 6^+} g(x) = \lim_{x \to 6^+} (x - 5) = 6 - 5 = 1$$

- Why can we plug in 6?

Because we are ***evaluating a limit*** and we can get as close to 6 as possible

Determining Limits Using Algebraic Properties of Limits

$$f(x) = \begin{cases} -\frac{3}{4}(x-3)+1, & x < 3 \\ \frac{1}{2}(x-3)-1, & x > 3 \end{cases}$$

I.E. WHAT HAPPENS IF WE WERE TO PLUG IN 3 FROM LEFT & RIGHT SIDES?

The function $f(x)$ is given above. Find the following values, if possible

$$\lim_{x \to 3^-} f(x) =$$

$$\lim_{x \to 3^+} f(x) =$$

$$\lim_{x \to 3} f(x) =$$

Another example - Determining Limits Using Algebraic Properties of Limits

$$k(x) = \begin{cases} x^2 - 1, & x < 2 \\ 4x - 7, & x \geq 2 \end{cases}$$

The function $k(x)$ is given above. Find the $\lim_{x \to 2} k(x)$ or show that it does not exist.

Next-> What Will We Learn?

- We can apply limit theorems when applicable.

- We can understand when theorems involving limits cannot be applied.

$$\lim_{x \to 0+} \frac{1}{x} \quad \to +\infty$$
$$\lim_{x \to 0-} \frac{1}{x} \quad \to -\infty$$
$$\lim_{x \to +\infty} \frac{1}{x} \quad \to 0^{+} = 0$$
$$\lim_{x \to -\infty} \frac{1}{x} \quad \to 0^{-} = 0$$

Theorems About Limits

If $\lim_{x \to c} f(x) = L \quad and \quad \lim_{x \to c} g(x) = M, \; then \; \ldots$

$\lim_{x \to c} [b f(x)] = b \cdot L$ (where b is a constant)	$\lim_{x \to c} [f(x) g(x)] = L \cdot M$
$\lim_{x \to c} [f(x) \pm g(x)] = L \pm M$	$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M} \, (when \; M \neq 0)$

Caution: If the conditions are NOT met, we CANNOT apply these theorems. We must find an alternative approach to investigate the limit.

Using Limit Laws

The limits exist, so we can use the theorems

$f(1) = -2$	$\lim_{x \to 1} f(x) = 3$
$g(1) = 5$	$\lim_{x \to 1} g(x) = -4$

Selected values and limits for the functions f and g are given above. Find the following:

A) $\lim_{x \to 1} [2f(x) + g(x)] =$

B) $\lim_{x \to 1} \frac{g(x)}{f(x)} =$

Theorems About Limits	
If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then ...	
$\lim_{x \to c} [bf(x)] = b \cdot L$ (where b is a constant)	$\lim_{x \to c} [f(x)g(x)] = L \cdot M$
$\lim_{x \to c} [f(x) \pm g(x)] = L \pm M$	$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ (when $M \neq 0$)
Caution: If the conditions are NOT met, we CANNOT apply these theorems. We must find an alternative approach to investigate the limit.	

Using Limit Laws

The limits exist, so we can use the theorems

$h(4) = -2$	$\lim_{x \to 4} h(x) = 7$
$k(4)$ is undefined	$\lim_{x \to 4} k(x) = -1$
$p(4) = 0$	$\lim_{x \to 4} p(x) = 0$

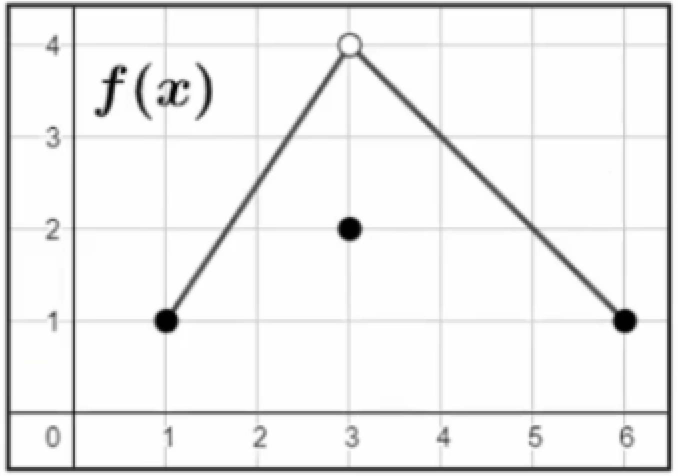
Selected values and limits for the functions h, k and p are given above. Find the following:

$$\lim_{x \to 4}[h(x) - 3k(x) + 2p(x)] =$$

Theorems About Limits	
If $\lim_{x \to c} f(x) = L$ *and* $\lim_{x \to c} g(x) = M$, *then* ...	
$\lim_{x \to c}[bf(x)] = b \cdot L$ (where b is a constant)	$\lim_{x \to c}[f(x)g(x)] = L \cdot M$
$\lim_{x \to c}[f(x) \pm g(x)] = L \pm M$	$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M} (when\ M \neq 0)$

Caution: If the conditions are NOT met, we CANNOT apply these theorems. We must find an alternative approach to investigate the limit.

Using Limit Laws



$g(3) = 5$	$\lim_{x \to 3} g(x) = 6$

The graph of function f and selected values and limits for the function g are shown above. Find the following:

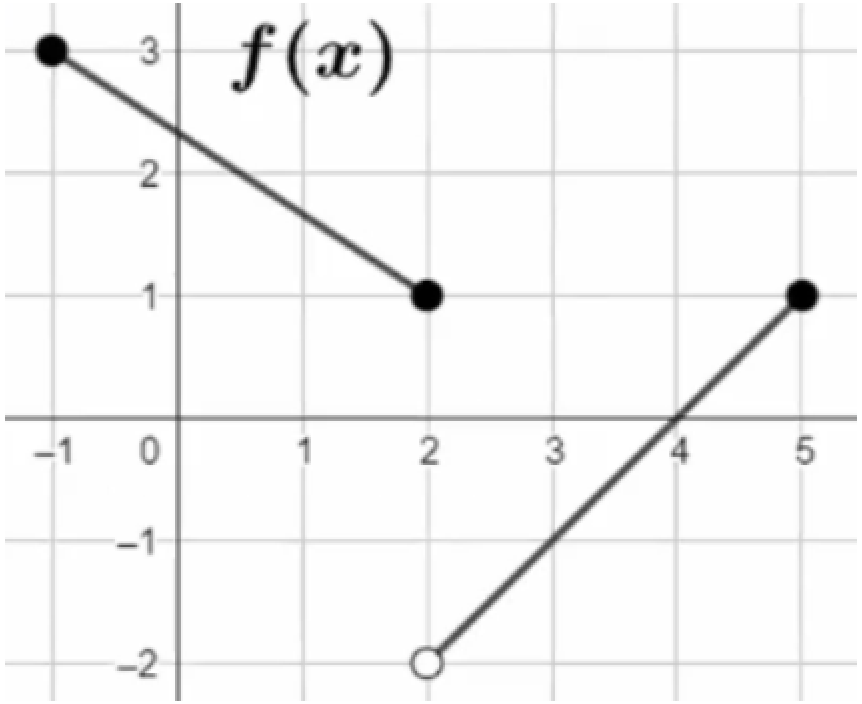
A) $\lim_{x \to 3} [f(x) + 2g(x)] =$

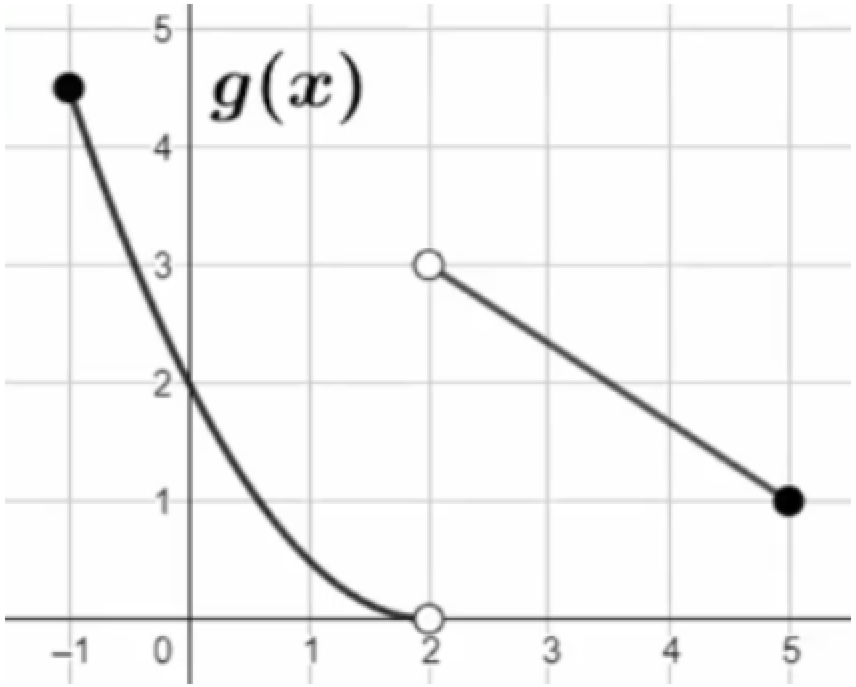
B) $\lim_{x \to 3} [g(x)(f(4) - f(x))] =$

- We can still evaluate many limits when traditional limit laws (theorems) do not apply.

- When traditional limit laws cannot be applied, we can investigate one sided limits to evaluate a limit.

Investigating One Sided Limits To Evaluate A Limit.



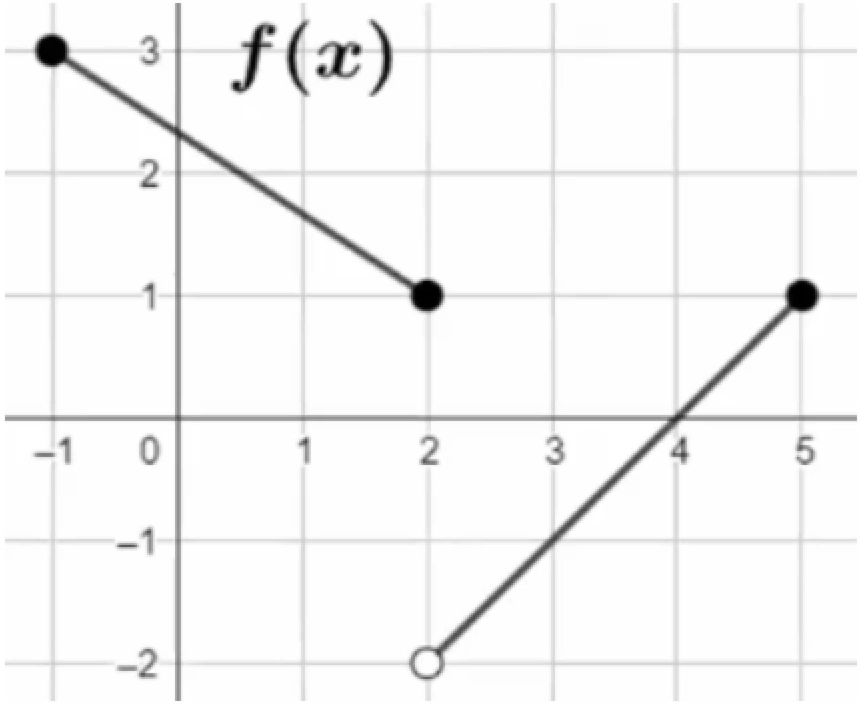


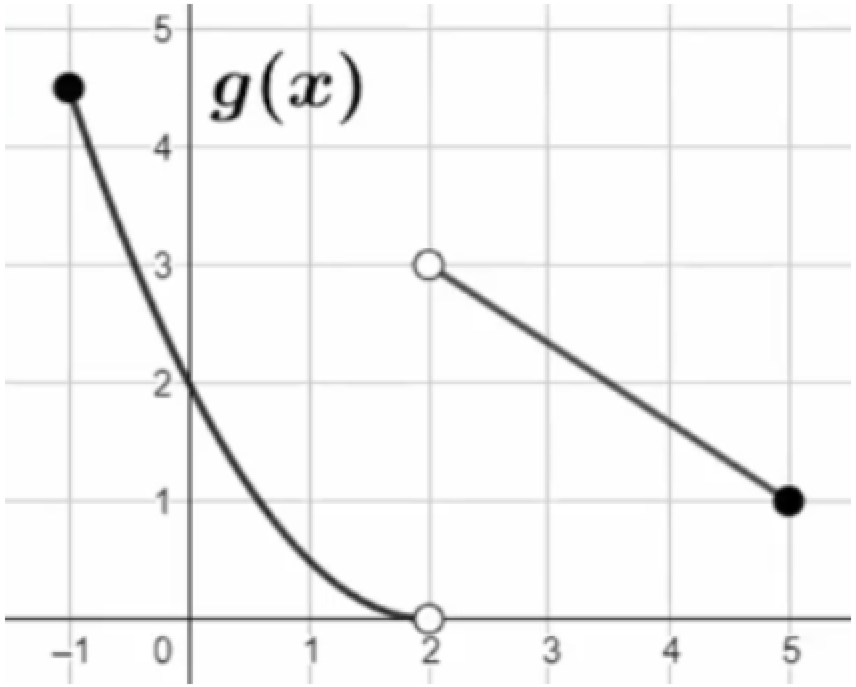
The graphs of $f(x)$ and $g(x)$ are above. Find $\lim_{x\to 2}[f(x)+g(x)]$ or show that it does not exist.

The traditional limit would say $\lim_{x\to 2}[f(x)+g(x)] = \lim_{x\to 2} f(x) + \lim_{x\to 2} g(x)$

BUT from the graph, limits of f and g does not exist…… THAT does not mean that the question has no limit.

Investigating One Sided Limits To Evaluate A Limit.





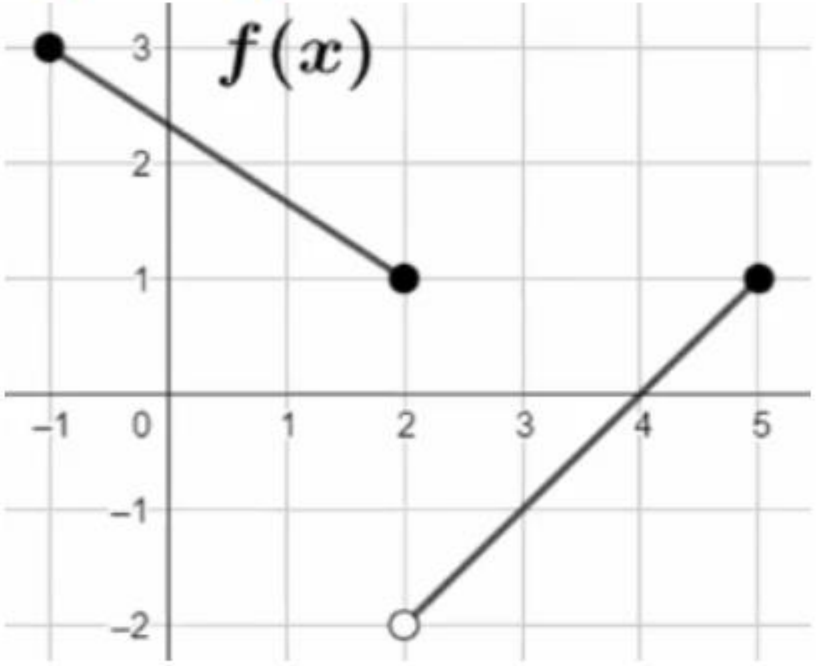
The graphs of $f(x)$ and $g(x)$ are above. Find $\lim_{x \to 2}[f(x) + g(x)]$ or show that it does not exist.

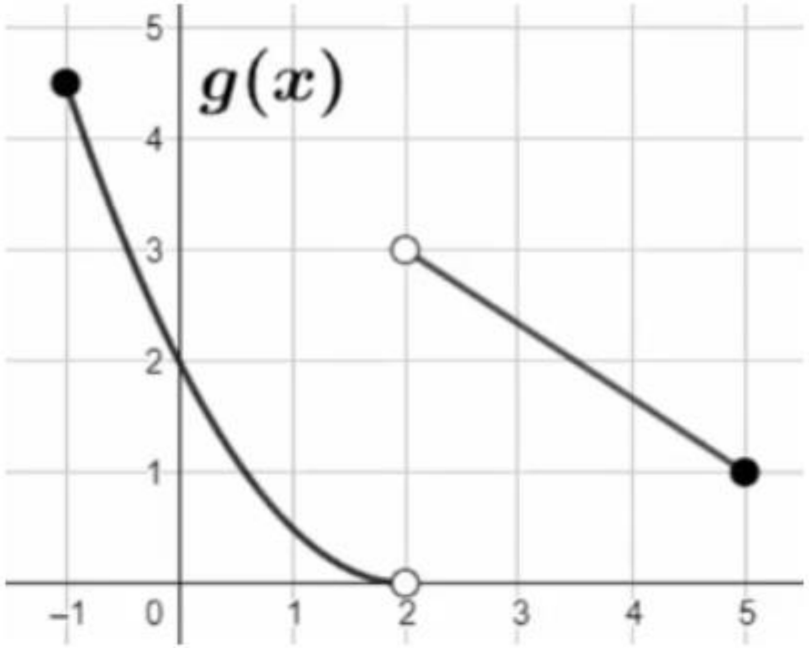
N.B. If the individual limit does not exist, it does not mean that there is no answer.

It just means that we can't use the traditional limit laws.

We need to investigate the problem as the way it was written from left and right sides.

Investigating One Sided Limits To Evaluate A Limit.





The graphs of $f(x)$ and $g(x)$ are above. Find $\lim_{x\to 2}[f(x)+g(x)]$ or show that it does not exist.

Left: $\lim_{x\to 2^-}[f(x)+g(x)] = \lim_{x\to 2^-} f(x) + \lim_{x\to 2^-} g(x) = 1 + 0 = 1$

Right: $\lim_{x\to 2^+}[f(x)+g(x)] = \lim_{x\to 2^+} f(x) + \lim_{x\to 2^+} g(x) = -2 + 3 = 1$

So: $\boldsymbol{lim_{x\to 2}[f(x)+g(x)] = 1}$

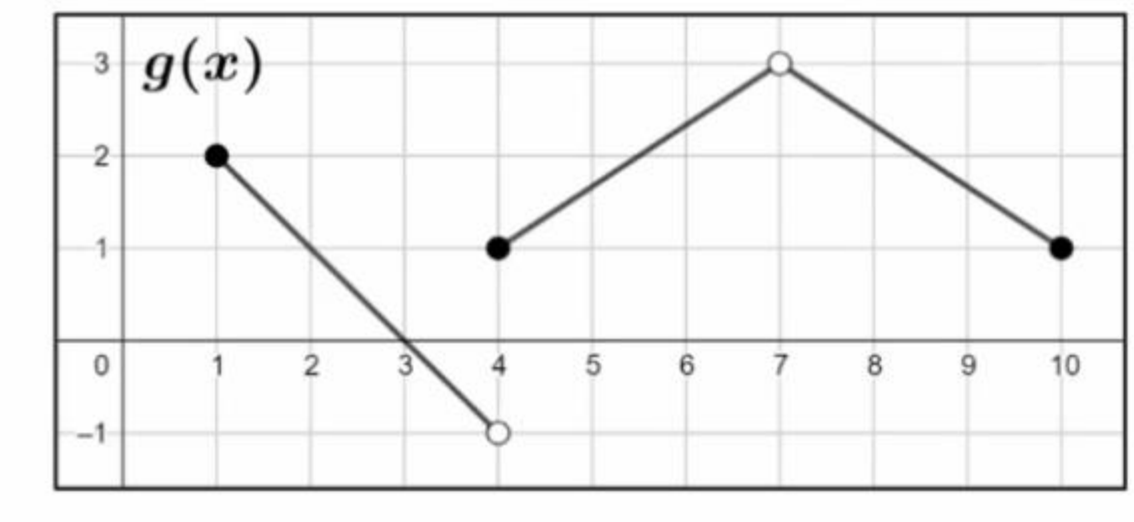
Investigating One Sided Limits To Evaluate A Limit.

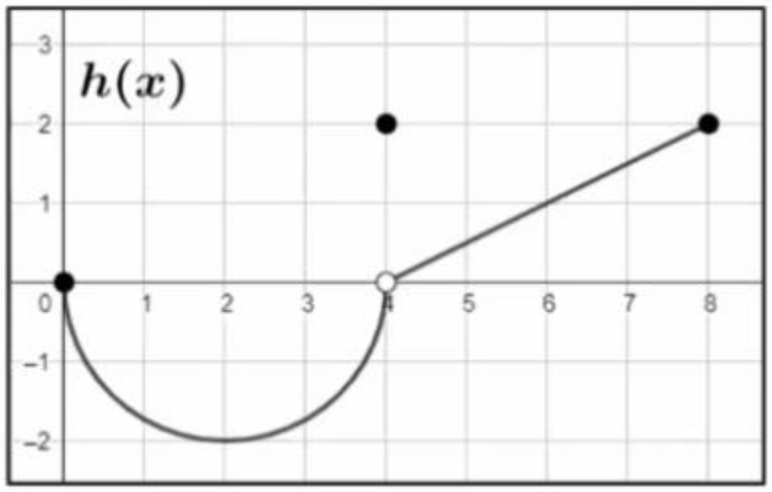
N.B.

- If you have a problem and you try to use the limit laws and one of them does not exist, don’t be too quick to say that the answer does not exist.

- **THINK ABOUT APPROACHING FROM THE LEFT AND THE RIGHT SIDES SEPARATELY AND SEE IF THEY APPROACH THE SAME VALUE.**

Another example: Advanced Limits: Products





The graphs of $g(x)$ and $h(x)$ are above. Find $\lim_{x \to 4}[g(x)h(x)]$ or show that it does not exist.

Summary : Theorems About Limits

If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then ...

$\lim_{x \to c} [bf(x)] = b \cdot L$ (where b is a constant)	$\lim_{x \to c} [f(x)g(x)] = L \cdot M$
$\lim_{x \to c} [f(x) \pm g(x)] = L \pm M$	$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M} (when\ M \neq 0)$

Caution: If the conditions are NOT met, we CANNOT apply these theorems. We must find an alternative approach to investigate the limit.