An estimator is **biased** if:

A. It is calculated incorrectly

B. Its sampling distribution is not normal

C. Its expected value is not equal to the true parameter

D. It has a large variance

C

A statistics student wants to estimate the average height of all students at their school. They take a random sample of 30 students and calculate the sample mean.

a) What is the **point estimate** of the population mean?

b) Explain what it means for this estimate to be **unbiased**.

c) Give an example of a way this estimate could become **biased**.

Stretch and Challenge Questions

1. Let $X_1, X_2, \ldots, X_n$ be a random sample with mean μ.
   Show whether $\bar{X}^2$ is an unbiased estimator of μ^2.

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2. Suppose $\hat{\theta}_1$ and $\hat{\theta}_2$ are both unbiased estimators of a parameter θ.
   Is $\frac{1}{2}(\hat{\theta}_1 + \hat{\theta}_2)$ always unbiased? Justify your answer.

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3. A sample variance is defined as

$$S_n^2 = \frac{1}{n}\sum(X_i - \bar{X})^2$$

Show that this estimator is biased for the population variance σ^2.

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4. Construct an estimator for μ of the form

$$a\bar{X}$$

Find the value of a such that the estimator is unbiased.

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1. **Estimator:** A statistic used to estimate a population parameter.
2. **Variance of an Estimator:** Measures the expected squared deviation from its mean; smaller variance → more precise estimate.
3. **Standard Error (SE):** The square root of the variance; used to quantify precision.
4. **Unbiased vs Biased:** Unbiased estimator has expected value equal to the parameter; biased does not.
5. **Mean Squared Error (MSE):** Combines variance and bias: $\mathrm{MSE} = \mathrm{Var} + \mathrm{Bias}^2$
6. **Sampling Distribution:** Shows how an estimator varies across samples; variance determines spread.
7. **Bias-Variance Trade-off:** Sometimes a slightly biased estimator is preferred if it has lower variance.
8. **Effect of Sample Size:** Increasing n reduces variance and SE, making estimates more precise.

A random sample of size n is taken from a population with mean μ and variance σ^2.

1. Define an estimator $\hat{\mu} = \bar{X}$. Show that $\hat{\mu}$ is unbiased.
2. Calculate the variance of $\hat{\mu}$ in terms of σ^2 and n.
3. A student proposes using $\hat{\mu}_2 = \frac{1}{2}X_1 + \frac{1}{2}X_2$ (where X_1, X_2 are the first two sample values) as an estimator of μ. Determine whether $\hat{\mu}_2$ is unbiased.
4. Explain how increasing the sample size n affects the variance and standard error of $\hat{\mu}$.
5. Define **mean squared error (MSE)** and explain its relationship with bias and variance.

Show that

$$S^2 = \frac{1}{\_\_\_\_} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

is an unbiased estimator of σ^2

🔷 Step 1: Key Identity

$$\sum (X_i - \bar{X})^2 = \sum (X_i - \_\_\_\_)^2 - n(\bar{X} - \_\_\_\_)^2$$

🔷 Step 2: Take Expectations

$$\mathbb{E}\left[\sum (X_i - \bar{X})^2\right] = \mathbb{E}\left[\sum (X_i - \_\_\_\_)^2\right] - \mathbb{E}\left[n(\bar{X} - \_\_\_\_)^2\right]$$

🔷 Step 3: Evaluate Each Term

$$\mathbb{E}\left[\sum (X_i - \mu)^2\right] = \_\_\_\_ \cdot \sigma^2$$

$$\mathbb{E}\left[n(\bar{X} - \mu)^2\right] = n \cdot \mathrm{Var}(\bar{X}) = n \cdot \frac{\sigma^2}{\_\_\_\_} = \_\_\_\_$$

🔷 Step 4: Combine Results

$$\mathbb{E}\left[\sum (X_i - \bar{X})^2\right] = \_\_\_\_\sigma^2 - \_\_\_\_ = (\_\_\_\_)\sigma^2$$