TURN AND TALK

1. Differentiate $y = (2x+3)^{-1}$.
2. Find $\frac{dy}{dx}$ for $y = (x^2 - 1)^{-1}$.
3. If $f(x) = 3x - 4$ and $f^{-1}(x)$ is its inverse, find $(f^{-1})'(x)$.

TURN AND TALK

1. $y = (2x+3)^{-1}$

$$\frac{dy}{dx} = -1(2x+3)^{-2} \cdot 2 = -\frac{2}{(2x+3)^2}$$

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2. $y = (x^2-1)^{-1}$

$$\frac{dy}{dx} = -1(x^2-1)^{-2} \cdot 2x = -\frac{2x}{(x^2-1)^2}$$

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3. $f(x) = 3x - 4$
   First find inverse: $f^{-1}(x) = \frac{x+4}{3}$

Derivative of inverse:

$$(f^{-1})'(x) = \frac{1}{3}$$

- Inverse Trigonometric Function — 反三角函数
- Arcsin — 反正弦函数
- Arccos — 反余弦函数
- Arctan — 反正切函数
- Domain — 定义域
- Range — 值域
- Principal Value — 主值
- Unit Circle — 单位圆
- One-to-One Function — 一一对应函数
- Horizontal Line Test — 水平线测试

UNIT 3 KNOWLEDGE - CALCULUS 12 – DIFFERENTIATION: COMPOSITE, IMPLICIT, & INVERSE FUNCTIONS

- **3.1** • The Chain Rule ✓
- **3.2** • Implicit Differentiation ✓
- **3.3** • Differentiating Inverse Functions ✓
- **3.4** Differentiating Inverse Trigonometric Functions
- **3.5** • Selecting Procedures for Calculating Derivatives
- **3.6** • Calculating Higher-Order Derivatives

What Will We Learn?

- How can we use implicit differentiation to develop rules for differentiating inverse trigonometric functions?

- How can we apply derivative rules with inverse trigonometric functions?

Inverse Trigonometric Derivative Rules

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1 + x^2}$$

Inverse Trigonometric Derivative Rules

Function	Derivative	Function	Derivative
$y = \sin^{-1}(u)$	$\frac{dy}{dx} = \frac{u'}{\sqrt{1-u^2}}$	$y = \cos^{-1}(u)$	$\frac{dy}{dx} = -\frac{u'}{\sqrt{1-u^2}}$
$y = \tan^{-1}(u)$	$\frac{dy}{dx} = \frac{u'}{1+u^2}$	$y = \cot^{-1}(u)$	$\frac{dy}{dx} = -\frac{u'}{1+u^2}$
$y = \sec^{-1}(u)$	$\frac{dy}{dx} = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$	$y = \csc^{-1}(u)$	$\frac{dy}{dx} = -\frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$

N.B: They are written in terms of u so we can think about u with Chain Rule because of possible inside functions.

Let’s see a proof and an example!

Proof: Differentiating an Inverse Trigonometric Function

Consider the function $y = \sin^{-1}(x)$. Find $\frac{dy}{dx}$

Steps	Solution
Recall:	
Step 1: Use trig. ratios $$sin\ (\theta) = \frac{opp}{hyp}$$	
Step 2: Take derivative using implicit differentiation & chain rule	

Proof: Differentiating an Inverse Trigonometric Function

Consider the function $y = \sin^{-1}(x)$. Find $\frac{dy}{dx}$

Steps	**Solution**
$\frac{dy}{dx} = \frac{1}{cos(y)}$ Step 3: Problem is in terms of x, so take the y-value & **re-express** it in terms of x using Δ.	
Step 4: Plug in $\cos(y)$	
Conclusion	

Recall: Trigonometric Derivative Formulas

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \cdot \tan x$$

$$\frac{d}{dx}[\csc x] = -\csc x \cdot \cot x$$

Proof: Differentiating an Inverse Trigonometric Function

The formula for the derivative of the arctangent function is derived in a similar way. If $y = \tan^{-1} x$, then $\tan y = x$. Differentiating $\tan y = x$ implicitly with respect to x, we have

Example 2- A Tangent Line Involving Inverse Tangent

Write the equation of the line tangent $f(x) = \tan^{-1}(3x)$ $when\ x = \frac{1}{3}$.

Steps	Solution
Step 1: Find the points (x, y)	
Step 2: Find slope	
Step 3: Write the equation	

Function	Derivative
$y = \sin^{-1}(u)$	$\frac{dy}{dx} = \frac{u'}{\sqrt{1-u^2}}$
$y = \tan^{-1}(u)$	$\frac{dy}{dx} = \frac{u'}{1+u^2}$

Next-> What Will We Learn?

- We’ll connect inverse trigonometric derivatives to previous topics and explore how these problems may appear.

Example 3- Find the instantaneous rate of change of the function $y = \tan^{-1}(4x)$ $when\ x = \frac{1}{2}$.

Steps	Solution
Step 1: Find u'	
Step 2: Find $\frac{dy}{dx}$	
Step 3: Find derivative at $x = \frac{1}{2}$	

Function	Derivative
$y = \sin^{-1}(u)$	$\frac{dy}{dx} = \frac{u'}{\sqrt{1-u^2}}$
$y = \tan^{-1}(u)$	$\frac{dy}{dx} = \frac{u'}{1+u^2}$

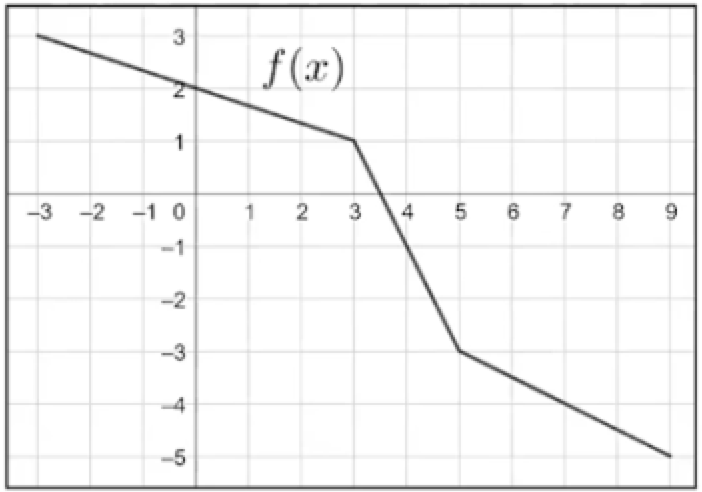
Example 4- $\lim_{h \to 0} \frac{\sin^{-1}\left(\frac{4}{5}+h\right) - \sin^{-1}\left(\frac{4}{5}\right)}{h}$

<table>
<tr><th>Steps</th><th>Solution</th></tr>
<tr><td>Recall limit definition of a derivative</td><td></td></tr>
<tr><td>Step 1: Write out $f(x)$ & find $f'(x)$
<table><tr><th>Function</th><th>Derivative</th></tr><tr><td>$y = \sin^{-1}(u)$</td><td>$\frac{dy}{dx} = \frac{u'}{\sqrt{1-u^2}}$</td></tr></table></td><td></td></tr>
<tr><td>Step 2: Find $f'\left(\frac{4}{5}\right)$</td><td></td></tr>
</table>

Find $g'(0)$

Steps	Solution
Step 1: Differentiate both sides to find $g'(x)$	$f(g(x)) = \tan^{-1}(1-x)$
Step 2: Plug in 0 to find $g'(0)$	

Function	Derivative
$y = \sin^{-1}(u)$	$\frac{dy}{dx} = \frac{u'}{\sqrt{1-u^2}}$
$y = \tan^{-1}(u)$	$\frac{dy}{dx} = \frac{u'}{1+u^2}$



Key Takeaways

Inverse Trigonometric Derivative Rules

Function	Derivative	Function	Derivative
$y = \sin^{-1}(u)$	$\frac{dy}{dx} = \frac{u'}{\sqrt{1-u^2}}$	$y = \cos^{-1}(u)$	$\frac{dy}{dx} = -\frac{u'}{\sqrt{1-u^2}}$
$y = \tan^{-1}(u)$	$\frac{dy}{dx} = \frac{u'}{1+u^2}$	$y = \cot^{-1}(u)$	$\frac{dy}{dx} = -\frac{u'}{1+u^2}$
$y = \sec^{-1}(u)$	$\frac{dy}{dx} = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$	$y = \csc^{-1}(u)$	$\frac{dy}{dx} = -\frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$

N.B: They are written in terms of u so we can think about u with Chain Rule because of possible inside functions.

TRUE or FALSE

1. **True or False:** The function $\sin^{-1}(x)$ returns an angle as its output.
2. **True or False:** The domain of $\tan^{-1}(x)$ is all real numbers.
3. **True or False:** Inverse trigonometric functions always have a unique output for every input in their domain.

TRUE or FALSE

1. **True** — $\sin^{-1}(x)$ (arcsin) gives an angle as its output.
2. **True** — $\tan^{-1}(x)$ is defined for all real numbers $x \in \mathbb{R}$.
3. **True** — Inverse trigonometric functions are defined to be one-to-one on restricted domains, so each input has a unique output.