PRECALCULUS · FUNCTIONS

Composite and Inverse Functions

Challenging review — one multi-part question, worked in full

4
parts, question first

2
supporting questions

8
true / false checks

Review before the mid-term. Mid-term topics: sets, quadratics, transformations, composite and inverse functions.

How this lesson runs

Key concept, worked example, questions to try — then true or false

Starter	One quick recall question on the shape of quadratic functions — attempt before the reveal.
Key points	Each section opens with the ideas you need, main point highlighted first, with formulae on their own slide.
Question	Four challenging parts appear on their own slides. Attempt each before turning the page.
Solution	The next slide gives the answer at 18pt, with a trap box naming the classic error.
True or false	The lesson ends with eight statements — one per slide, answer on the slide that follows.

Key topic

What this lesson is about, in one page

The skill	Build composite functions $(f \circ g)(x)$ and track the domain at every stage of the composition.
The inverse	Swap x and y, solve, and state the domain and range of f^{-1} from those of f.
The twist	For the restricted function $h(x) = x^2 - 2$, check that $h(h^{-1}(x)) = x$ on the restricted domain.
The challenge	Solve $f(x) = f^{-1}(x)$. The quadratic that appears has a negative discriminant — the honest answer is that there is no real solution.

Vocabulary

Key terms with Chinese translations

composition	**复合**	applying one function to the output of another, $(f \circ g)(x)$
inner function	**内函数**	$g(x)$, the function applied first
outer function	**外函数**	$f(x)$, the function applied second
one-to-one	**一一对应**	each output comes from exactly one input — needed for an inverse
inverse function	**反函数**	$f^{-1}(x)$, undoes f: domain and range swap
domain	**定义域**	the set of allowable inputs of a function
range	**值域**	the set of all outputs of a function
horizontal line test	**水平线检验**	a graph test for one-to-one: no horizontal line cuts the curve twice
restricted domain	**限定定义域**	limiting the domain (e.g. $x \geq 0$) to make a function one-to-one
rational function	**有理函数**	a quotient of polynomials, undefined where the denominator is 0

S

Starter · The shape of quadratic functions

What you must remember before composing functions

- One quick recall question
- Vertex, axis of symmetry and the role of a

Key points · the shape of a quadratic

Everything about $y = ax^2 + bx + c$ in five lines

1	**The parabola**	Every quadratic graph is a parabola. $a > 0$ opens upward; $a < 0$ opens downward.
2	**Axis of symmetry**	The mirror line $x = -b/(2a)$. The vertex sits on it.
3	**Vertex**	At $x = -b/(2a)$. It is the minimum when $a > 0$ and the maximum when $a < 0$.
4	**Roots (zeros)**	Solve $ax^2 + bx + c = 0$. The discriminant $b^2 - 4ac$ decides: two, one or no real roots.
5	**y-intercept**	The curve crosses the y-axis at $(0, c)$.

Question S1

5 marks

Starter · Shape of quadratics

THE QUESTION

For the quadratic $f(x) = 2x^2 + 4x - 6$:

(a) Does the parabola open upward or downward? How do you know?
(b) Find the equation of the axis of symmetry.
(c) Find the coordinates of the vertex.
(d) Find the y-intercept.
(e) Solve $f(x) = 0$ and state the roots.

Question S1 — solution

5 marks

Starter · Shape of quadratics

ANSWERS ONLY

- (a) $a = 2 > 0$, so it opens upward and the vertex is a minimum.
- (b) $x = -4/(2\cdot 2) = -1$.
- (c) $f(-1) = 2 - 4 - 6 = -8$, so the vertex is $(-1, -8)$.
- (d) $(0, -6)$.
- (e) $2x^2 + 4x - 6 = 2(x + 3)(x - 1)$, so the roots are $x = -3$ and $x = 1$.

TRAP Signs in $-b/(2a)$: with $b = 4$ the axis is NEGATIVE, $x = -1$. Sign slips here cost the vertex mark too.

01

The Challenging Question

Composite and inverse functions in four parts

- $f(x) = (2x - 5)/(x + 3)$, $g(x) = \sqrt{(x - 1)}$, $h(x) = x^2 - 2$ for $x \geq 0$
- Key points and formulae first — then parts (a) to (d), each question first

Key points · composite and inverse functions

Four ideas that unlock all four parts

1	**Inside first**	$(f \circ g)(x) = f(g(x))$: apply g first, then f. Work from the inside out.
2	**Domain of a composite**	x must be in the domain of g AND g(x) must be in the domain of f. Check both stages.
3	**Finding f^{-1}**	Swap x and y, solve for y. The domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.
4	**Restricted inverses**	$h(x) = x^2 - 2$ is not one-to-one on all reals — but on $x \geq 0$ it is, and $h^{-1}(x) = \sqrt{(x + 2)}$.

Formulae to use

Everything you need on one page — no calculator formulae here

Composition	$(f \circ g)(x) = f(g(x))$ — apply g first, then f
Inverse recipe	$y = f(x)$ → swap x and y → solve for y
Domain swap	domain of f^{-1} = range of f; range of f^{-1} = domain of f
Rational domain	a quotient is undefined where its denominator equals 0
Restricted square root	$h(x) = x^2 - 2$ on $x \geq 0$ $\Leftrightarrow$ $h^{-1}(x) = \sqrt{(x + 2)}$, domain $x \geq -2$
Quadratic discriminant	for $ax^2 + bx + c = 0$: $\Delta = b^2 - 4ac$ — $\Delta < 0$ means no real roots

Question (a)

3 marks

Composite and inverse · part (a)

THE QUESTION

$f(x) = (2x - 5)/(x + 3)$, $g(x) = \sqrt{(x - 1)}$.

(a) Find $(f \circ g)(x)$ in simplified form, and state its domain.

Show how the domain follows from BOTH functions.

Question (a) — solution

3 marks

Composite and inverse · part (a)

ANSWERS ONLY

- $(f \circ g)(x) = (2\sqrt{(x - 1)} - 5)/(\sqrt{(x - 1)} + 3)$
- Domain: $x \geq 1$
- Reason: g needs $x - 1 \geq 0$; the denominator $\sqrt{(x - 1)} + 3$ is strictly positive for every $x \geq 1$, so no further restriction.

TRAP TRAP: the denominator $\sqrt{(x - 1)} + 3$ can NEVER be 0 — a square root is never negative. The domain is $x \geq 1$, not $x > 10$.

Question (b)

4 marks

Composite and inverse · part (b)

THE QUESTION

$f(x) = (2x - 5)/(x + 3)$.

(b) Find $f^{-1}(x)$ in simplified form, and state its domain and range.

Question (b) — solution

4 marks

Composite and inverse · part (b)

ANSWERS ONLY

- $f^{-1}(x) = (-3x - 5)/(x - 2)$
- Domain of f^{-1}: $x \neq 2$
- Range of f^{-1}: $y \neq -3$

TRAP TRAP: swap x and y BEFORE solving. And f^{-1} is a function of x — the denominator zero x = 2 excludes one input, it does not make the whole inverse undefined.

Question (c)

3 marks

Composite and inverse · part (c)

THE QUESTION

$h(x) = x^2 - 2$ for $x \geq 0$.

(c) Find $h^{-1}(x)$ and show that $h(h^{-1}(x)) = x$ on the restricted domain. State the domain on which the identity holds.

Question (c) — solution

3 marks

Composite and inverse · part (c)

ANSWERS ONLY

- $h^{-1}(x) = \sqrt{(x + 2)}$, domain $x \geq -2$
- $h(h^{-1}(x)) = (\sqrt{(x + 2)})^2 - 2 = x + 2 - 2 = x$
- Valid for every $x \geq -2$ (the domain of h^{-1}), since $h^{-1}(x) = \sqrt{(x + 2)} \geq 0$ is inside h's restricted domain.

TRAP TRAP: the identity only holds because the domain is restricted. Without $x \geq 0$, h would be two-to-one and no inverse would exist.

Question (d)

4 marks

Composite and inverse · part (d)

THE QUESTION

$f(x) = (2x - 5)/(x + 3)$, $f^{-1}(x) = (-3x - 5)/(x - 2)$.

(d) Solve $f(x) = f^{-1}(x)$.

The conclusion may surprise you — check it carefully before you commit.

Question (d) — solution

4 marks

Composite and inverse · part (d)

ANSWERS ONLY

- Cross-multiplying: $(2x - 5)(x - 2) = (-3x - 5)(x + 3)$
- $2x^2 - 9x + 10 = -3x^2 - 14x - 15$
- $5x^2 + 5x + 25 = 0$, i.e. $x^2 + x + 5 = 0$
- $\Delta = 1^2 - 4 \cdot 5 = -19 < 0$
- Conclusion: NO real solutions

TRAP TRAP: do not stop at the quadratic. The discriminant is negative, so $f(x) = f^{-1}(x)$ has NO real solution — and $f(x) = x$ would give the same verdict. A wrong 'solution' here loses all 4 marks.

02

Supporting Practice

Two shorter questions to secure the method

- Q1: composites in both orders, plus an inverse
- Q2: inverse of a square-root function

Question P1

4 marks

Supporting practice · composites

THE QUESTION

$f(x) = x/(x - 1)$, $g(x) = x^2 + 3$.

(a) Find $(g \circ f)(x)$ and $(f \circ g)(x)$ in simplified form.
(b) Find $f^{-1}(x)$ and state its domain.
(c) Confirm that $g \circ f \neq f \circ g$ in general.

Question P1 — solution

4 marks

Supporting practice · composites

ANSWERS ONLY

- (a) $(g \circ f)(x) = x^2/(x-1)^2 + 3$; $(f \circ g)(x) = (x^2+3)/(x^2+2)$
- (b) $f^{-1}(x) = x/(x-1)$ — f is self-inverse! Domain: $x \neq 1$
- (c) At $x = 2$: $(g \circ f)(2) = 7$, $(f \circ g)(2) = 7/6$ — different, so order matters.

TRAP TRAP: $f^{-1}(x) = x/(x-1)$ looks identical to f — check it: it really is self-inverse. Do not 'correct' it into something else.

Question P2

3 marks

Supporting practice · square-root inverse

THE QUESTION

$k(x) = \sqrt{(x - 4)} + 2$, with domain $x \geq 4$.

(a) Explain why k is one-to-one.

(b) Find $k^{-1}(x)$ and state its domain and range.

Question P2 — solution

3 marks

Supporting practice · square-root inverse

ANSWERS ONLY

- (a) $\sqrt{(x - 4)}$ is strictly increasing on $x \geq 4$, so k passes the horizontal line test.
- (b) $k^{-1}(x) = (x - 2)^2 + 4$
- Domain of k^{-1}: $x \geq 2$ (the range of k); Range of k^{-1}: $y \geq 4$ (the domain of k).

TRAP TRAP: the inverse of a square root is a SQUARE. Its domain comes from the RANGE of k, not from solving $x - 2 \geq 0$.

Mark scheme

Marks by question, with what earns each mark

S1	**5**	1 per part: direction, axis, vertex, intercept, roots
(a)	**3**	1 composition formed · 1 simplified · 1 domain $x \geq 1$ justified
(b)	**4**	1 swap-and-solve method · 1 f^{-1} correct · 1 domain $x \neq 2$ · 1 range $y \neq -3$
(c)	**3**	1 $h^{-1} = \sqrt{(x + 2)}$ with domain · 1 identity shown · 1 valid-domain statement
(d)	**4**	1 cross-multiply · 1 quadratic formed · 1 discriminant evaluated · 1 'no real solutions' conclusion
P1	**4**	1 per composite · 1 inverse with domain · (c) covered by counter-example line
P2	**3**	1 one-to-one reason · 1 inverse · 1 domain and range swapped correctly

Main question total: 14 marks. Supporting practice: 7 marks. Starter: 5 practice marks (not assessed).

Answer key

One line per question

ANSWERS

- S1 up; axis $x = -1$; vertex $(-1, -8)$; $(0, -6)$; roots -3 and 1
- (a) $(f \circ g)(x) = (2\sqrt{(x-1)} - 5)/(\sqrt{(x-1)} + 3)$; domain $x \geq 1$
- (b) $f^{-1}(x) = (-3x - 5)/(x - 2)$; domain $x \neq 2$, range $y \neq -3$
- (c) $h^{-1}(x) = \sqrt{(x + 2)}$; $h(h^{-1}(x)) = x$ for $x \geq -2$
- (d) $5x^2 + 5x + 25 = 0$, $\Delta = -19 \rightarrow$ no real solutions
- P1 $g \circ f = x^2/(x-1)^2 + 3$; $f \circ g = (x^2+3)/(x^2+2)$; $f^{-1} = x/(x-1)$, $x \neq 1$

QUICK VERIFICATION

- P2 $k^{-1}(x) = (x - 2)^2 + 4$; domain $x \geq 2$, range $y \geq 4$
- Check (b): $f(f^{-1}(x)) = x$ for every $x \neq 2$.
- Check (d): any real solution would need $x^2 + x + 5 = 0$,
- but its minimum value is $4.75 > 0$ — no real roots, as $\Delta < 0$.
- Domain rule: restrictions travel from the inside out —
- g's domain gates the composite, f's gate is never triggered here.

True or false · 1

T / F

Decide, then justify your decision — answer on the next slide

For any two functions f and g, (f∘g)(x) = (g∘f)(x).

True or false · 1 — answer

FALSE

FALSE

Order matters. With $f(x) = x/(x - 1)$ and $g(x) = x^2 + 3$ we saw $(g \circ f)(2) = 7$ but $(f \circ g)(2) = 7/6$. Equality only holds for special pairs.

True or false · 2

T / F

Decide, then justify your decision — answer on the next slide

The domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

True or false · 2 — answer

TRUE

TRUE

Exactly. Inverting swaps the roles of inputs and outputs — this is how you state the domain and range of any inverse.

True or false · 3

T / F

Decide, then justify your decision — answer on the next slide

Every function has an inverse function.

True or false · 3 — answer

FALSE

FALSE

Only one-to-one functions do. $y = x^2$ maps both -2 and 2 to 4, so undoing is impossible unless the domain is restricted.

True or false · 4

T / F

Decide, then justify your decision — answer on the next slide

If f is one-to-one, then $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}.

True or false · 4 — answer

TRUE

TRUE

That is the definition of an inverse: apply f^{-1}, then f, and you are back where you started — for every input of f^{-1}.

True or false · 5

T / F

Decide, then justify your decision — answer on the next slide

The domain of f∘g must take into account the domain of g.

True or false · 5 — answer

TRUE

TRUE

Yes — and also whether g(x) lands in the domain of f. In part (a), g's $x \geq 1$ was the only restriction that bit.

True or false · 6

T / F

Decide, then justify your decision — answer on the next slide

A rational function is undefined at the zeros of its denominator.

True or false · 6 — answer

TRUE

TRUE

Division by 0 is undefined. For $f(x) = (2x - 5)/(x + 3)$, $x = -3$ is excluded; for $f^{-1}(x)$, $x = 2$ is excluded.

True or false · 7

T / F

Decide, then justify your decision — answer on the next slide

$f^{-1}(x)$ can be found by swapping x and y and solving for y.

True or false · 7 — answer

TRUE

TRUE

That is the standard recipe, and it works whenever f is one-to-one on the domain being used.

True or false · 8

T / F

Decide, then justify your decision — answer on the next slide

If $(f \circ g)(x) = f(x)$ for all x, then g must be the identity function $g(x) = x$.

True or false · 8 — answer

FALSE

FALSE

Not necessarily. If f is not one-to-one, a different g can hide inside f. Composition does not let you 'cancel' f unless f is one-to-one.

FIVE THINGS TO REMEMBER

Do these and composite and inverse questions will not cost you marks

- Compose from the inside out, and check BOTH domains — the inner one always gates the composite.
- Swap x and y first, then solve, to find an inverse — and state its domain from the range of f.
- A restricted domain ($x \geq 0$) is what makes $y = x^2$ invertible: $h^{-1}(x) = \sqrt{(x + 2)}$ lives on $x \geq -2$.
- A rational function's domain excludes every zero of its denominator — but $\sqrt{(x - 1)} + 3$ is never 0.
- Solve $f(x) = f^{-1}(x)$ honestly: here the discriminant is −19, so the answer is 'no real solutions'.

Matching worksheet: Composite and Inverse Review Worksheet — 18 marks, 55 minutes.