PRECALCULUS · POLYNOMIALS

Algebraic Division

Long division, the remainder theorem and the factor theorem

1
starter: quadratics

1
method: long division

11
questions worked

Built from the 2-04 Dividing Polynomials booklet

How this lesson runs

Key concept, worked example, questions to try — then true or false

Starter	Three quick recall questions on the shape of quadratic functions — attempt before the reveal.
Key points	Each section opens with a summary of the ideas you need, main point highlighted first.
Question	Questions appear on their own slides. Attempt each before turning the page.
Solution	The next slide gives the condensed solution with the working set out line by line.
True or false	The lesson ends with eight statements — one per slide, answer on the slide that follows.

Key topic · Algebraic division

What this lesson is about, in one page

The skill	Divide one polynomial by another using long division — the same four steps as arithmetic long division.
The structure	Dividend = Divisor × Quotient + Remainder. Every answer can be checked by rebuilding the dividend.
The remainder theorem	The remainder when f(x) is divided by (x − k) is exactly f(k) — no division needed.
The factor theorem	If f(k) = 0, then (x − k) is a factor of f(x), and k is a root of f(x) = 0.

Vocabulary

Key terms with Chinese translations

dividend	□□□	the polynomial being divided, f(x)
divisor	□□	the polynomial you divide by, d(x)
quotient	□	the result on top, q(x)
remainder	□□	what is left over, r(x)
factor	□□	d(x) is a factor if the remainder is 0
root (zero)	□□□□□	a value of x with f(x) = 0
degree	□□	the highest power of x
coefficient	□□	the number in front of each power
long division	□□□	the four-step division method
polynomial	□□□	a sum of terms like ax^n

S

Starter · The shape of quadratic functions

What you must remember before dividing

- Two quick recall questions
- Vertex, axis of symmetry, roots and the role of a

Key points · the shape of a quadratic

Everything about $y = ax^2 + bx + c$ in five lines

1	**The parabola**	Every quadratic graph is a parabola. $a > 0$ opens upward (a smile); $a < 0$ opens downward (a frown).
2	**Axis of symmetry**	The mirror line $x = -b/(2a)$. The vertex sits on it.
3	**Vertex**	At $x = -b/(2a)$. It is the minimum when $a > 0$ and the maximum when $a < 0$.
4	**Roots (zeros)**	Where the curve crosses the x-axis: solve $ax^2 + bx + c = 0$. Two, one (repeated) or none — see the discriminant $b^2 - 4ac$.
5	**y-intercept**	The curve crosses the y-axis at $(0, c)$.

Question S1

5 marks

Starter · Shape of quadratics

THE QUESTION

For the quadratic $f(x) = 2x^2 - 8x + 6$:

(a) Does the parabola open upward or downward? How do you know?
(b) Find the equation of the axis of symmetry.
(c) Find the coordinates of the vertex.
(d) Find the y-intercept.
(e) Solve $f(x) = 0$ and state the roots.

Question S1 — solution

5 marks

Starter · Shape of quadratics

CONDENSED SOLUTION

- (a) Upward. $a = 2 > 0$, so the parabola is a “smile” and the vertex is a minimum.
- (b) Axis of symmetry: $x = -b/(2a) = -(-8)/(2\cdot 2) = 8/4 = 2$, so $x = 2$.
- (c) Vertex: $f(2) = 2(4) - 16 + 6 = -2$, so the vertex is $(2, -2)$.
- (d) y-intercept: $(0, c) = (0, 6)$.
- (e) $2x^2 - 8x + 6 = 2(x^2 - 4x + 3) = 2(x - 1)(x - 3)$, so the roots are $x = 1$ and $x = 3$.
- Check: the roots are symmetric about the axis $x = 2$. ✓

TRAP Signs in $-b/(2a)$. With $b = -8$ the minus signs cancel: $-(-8) = +8$.

Question S2

4 marks

Starter · Shape of quadratics

THE QUESTION

The parabola $y = -x^2 + 4x - 3$ is drawn on the axes.

(a) State the direction of opening and the coordinates of the vertex.
(b) State the equation of the axis of symmetry.
(c) Factorise and write down the roots.
(d) Sketch the shape, labelling the vertex, both roots and the y-intercept.

Question S2 — solution

4 marks

Starter · Shape of quadratics

CONDENSED SOLUTION

- (a) $a = -1 < 0$, so it opens downward (a “frown”) and the vertex is a maximum.
- Axis: $x = -4/(2\cdot(-1)) = 2$. Vertex: $y = -4 + 8 - 3 = 1$, so $(2, 1)$.
- (c) $-x^2 + 4x - 3 = -(x^2 - 4x + 3) = -(x - 1)(x - 3)$, so the roots are $x = 1$ and $x = 3$.
- (d) Sketch: downward parabola through $(0, -3)$, $(1, 0)$, vertex $(2, 1)$, $(3, 0)$.
- Why this matters today: dividing a polynomial by $(x - k)$ connects directly to these roots — a zero remainder means k is a root of the cubic.

TRAP Read the sign of a from the coefficient in FRONT of x^2, not from the constant term.

01

The Division Algorithm

What dividing polynomials actually produces

- Dividend = Divisor × Quotient + Remainder
- The language used in every mark scheme

Key points · the division algorithm

Four names you must use correctly

1	**The identity**	$f(x) = d(x)\cdot q(x) + r(x)$, where deg $r(x)$ < deg $d(x)$ or $r(x) = 0$.
2	**Dividend**	$f(x)$, the polynomial being divided. It goes inside the division frame.
3	**Divisor**	$d(x)$, the polynomial you divide by. It sits outside the frame.
4	**Quotient and remainder**	$q(x)$ sits on top; $r(x)$ is what is left over at the bottom. Remainder 0 means $d(x)$ divides evenly.

Question A1

2 marks

The division algorithm

THE QUESTION

In the division $(x^3 + 4x^2 - 3x + 10) \div (x + 2)$:

(a) State the dividend and the divisor.
(b) The result is $x^2 + 2x - 7$ with remainder 24. Write the division in the form $f(x) = d(x) \cdot q(x) + r(x)$.
(c) State the degree of the quotient $q(x)$ and explain how you know.

Question A1 — solution

2 marks

The division algorithm

CONDENSED SOLUTION

- (a) Dividend $f(x) = x^3 + 4x^2 - 3x + 10$; divisor $d(x) = x + 2$.
- (b) $x^3 + 4x^2 - 3x + 10 = (x + 2)(x^2 + 2x - 7) + 24$.
- (c) deg q = 2. Degrees subtract: deg q = deg f − deg d = 3 − 1 = 2.

TRAP Always check: divisor × quotient + remainder must rebuild the dividend exactly. Use it to verify your answer.

02

Long division

The four-step cycle: divide, multiply, subtract, bring down

- Missing terms must be written in with zero coefficients
- A worked example, then questions to try

Key points · polynomial long division

Same four steps as arithmetic long division

1 **The cycle** — DIVIDE the leading terms, MULTIPLY the whole divisor by that answer, SUBTRACT, then bring down the next term. Repeat.

2 **Missing terms** — Write the dividend in descending powers and insert 0·(any missing power): $y^4 + 2y^2 - y + 5$ becomes $y^4 + 0y^3 + 2y^2 - y + 5$.

3 **Subtract carefully** — Subtracting changes EVERY sign in the row. $(a - b) \neq (a + b)$.

4 **Stop when** — The remainder has degree less than the divisor. Then write the answer as quotient + remainder/divisor.

Worked example · long division

follow it

$(y^4 + 2y^2 - y + 5) \div (y^2 - y + 1)$

SET-UP

$$y^4 + 0y^3 + 2y^2 - y + 5$$

divided by $y^2 - y + 1$

Note the inserted $0y^3$ term: without it the columns slip and every later step lands in the wrong place.

The four steps, over and over:

- 1. DIVIDE the first terms: $y^4 \div y^2 = y^2$
- 2. MULTIPLY: $y^2(y^2 - y + 1) = y^4 - y^3 + y^2$
- 3. SUBTRACT (flip every sign)
- 4. BRING DOWN the next term

THE WORKING

$$\begin{array}{rll} & y^2 + y + 2 & \\ y^2-y+1 \;\searrow\; y^4 + 0y^3 + 2y^2 - y + 5 & & \\ y^4 - y^3 + y^2 & & y^4 \div y^2 = y^2 \\ y^3 + y^2 & \downarrow & \\ y^3 - y^2 + y & & y^3 \div y^2 = y \\ 2y^2 - 2y + 5 & & \\ 2y^2 - 2y + 2 & & 2y^2 \div y^2 = 2 \\ 3 & & \text{remainder } 3 \end{array}$$

Answer: $y^2 + y + 2 + 3/(y^2 - y + 1)$

Try it yourself

your turn

Attempt before turning the page — answer on the next slide

YOUR TURN

Use long division to divide:

$(x^3 + 2x^2 - 5x + 6) \div (x + 1)$

Work the cycle: divide, multiply, subtract, bring down. Write your answer as q(x) + r(x)/d(x).

Question Y1 — solution

3 marks

Try it · long division

CONDENSED SOLUTION

- No missing terms: $x^3 + 2x^2 - 5x + 6$.
- $x^3 \div x = x^2$. Multiply: $x^3 + x^2$. Subtract: x^2 remains; bring down $-5x$.
- $x^2 \div x = x$. Multiply: $x^2 + x$. Subtract: $-6x$ remains; bring down 6.
- $-6x \div x = -6$. Multiply: $-6x - 6$. Subtract: 12.
- Quotient $x^2 + x - 6$, remainder 12.
- Answer: $x^2 + x - 6 + 12/(x + 1)$. Check: $(x+1)(x^2+x-6)+12 = x^3+2x^2-5x+6$ ✓

TRAP Subtraction flips every sign: $-5x - (+x) = -6x$.

Question L1

3 marks

Long division

THE QUESTION

Use long division to find the quotient and remainder when

$(x^3 + 4x^2 - 3x + 10) \div (x + 2)$

Write your answer in the form $q(x) + r(x)/d(x)$.

Question L1 — solution

3 marks

Long division

CONDENSED SOLUTION

- Set up in descending powers (no terms missing here): $x^3 + 4x^2 - 3x + 10$.
- $x^3 \div x = x^2$. Multiply: $x^2(x + 2) = x^3 + 2x^2$. Subtract: $2x^2$ remains; bring down $-3x$.
- $2x^2 \div x = 2x$. Multiply: $2x^2 + 4x$. Subtract: $-7x$ remains; bring down 10.
- $-7x \div x = -7$. Multiply: $-7x - 14$. Subtract: 24.
- Quotient $x^2 + 2x - 7$, remainder 24.
- Answer: $x^2 + 2x - 7 + 24/(x + 2)$.

TRAP Subtraction flips every sign: $-3x - (+4x) = -7x$. Sign slips here are the most common way to lose the method marks.

Question L2

3 marks

Long division

THE QUESTION

Use long division to divide $(2x^3 - 3x^2 + 4x - 5)$ by $(x - 2)$.

Write your answer in the form $q(x) + r(x)/d(x)$, and check your answer by rebuilding the dividend.

Question L2 — solution

3 marks

Long division

CONDENSED SOLUTION

- $2x^3 \div x = 2x^2$. Multiply: $2x^3 - 4x^2$. Subtract: x^2 remains; bring down 4x.
- $x^2 \div x = x$. Multiply: $x^2 - 2x$. Subtract: 6x remains; bring down −5.
- $6x \div x = 6$. Multiply: $6x - 12$. Subtract: 7.
- Quotient $2x^2 + x + 6$, remainder 7.
- Answer: $2x^2 + x + 6 + 7/(x - 2)$.
- Check: $(x - 2)(2x^2 + x + 6) + 7 = 2x^3 - 3x^2 + 4x - 5$ ✓

TRAP A negative remainder is fine. If yours is positive when it should be negative, re-check the subtraction rows.

Question L3

3 marks

Long division

THE QUESTION

Use long division to divide $(3x^4 - 5x^2 + 2x - 1)$ by $(x^2 + 1)$.

Remember to insert any missing terms before you start.

Question L3 — solution

3 marks

Long division

CONDENSED SOLUTION

- Insert the missing term: $3x^4 + 0x^3 - 5x^2 + 2x - 1$.
- $3x^4 \div x^2 = 3x^2$. Multiply: $3x^4 + 3x^2$. Subtract: $-8x^2$; bring down $2x$.
- $-8x^2 \div x^2 = -8$. Multiply: $-8x^2 - 8$. Subtract: $2x + 7$.
- Quotient $3x^2 - 8$, remainder $2x + 7$.
- Answer: $3x^2 - 8 + (2x + 7)/(x^2 + 1)$. The remainder is a polynomial — allowed, since its degree (1) is less than the divisor's degree (2).

TRAP The remainder can be a polynomial. You stop when its degree drops below the divisor's degree, not when it stops being non-zero.

03

The remainder theorem and the factor theorem

What the remainder tells you about roots

- The remainder theorem: remainder = f(k)
- The factor theorem: f(k) = 0 means (x − k) is a factor

Key points · remainders and factors

Two theorems that turn division into root-finding

1. **Remainder theorem** — The remainder when f(x) is divided by (x − k) is exactly f(k). No division needed.
2. **Factor theorem** — If f(k) = 0, then (x − k) is a factor of f(x) — and k is a root of the equation f(x) = 0.
3. **Reading it back** — A zero remainder in long division means k is a root: the divisor divides evenly.
4. **Watch the sign** — For divisor (x + 3), k = −3. The remainder is f(−3), not f(3).

Worked example · the two theorems

follow it

$f(x) = x^3 - 2x^2 + 5x - 7$ divided by $(x - 3)$

BY LONG DIVISION

$$\begin{array}{rl} & x^2 + x + 8 \\ x-3 \;\big\lfloor\!\!\big\rfloor\; & x^3 - 2x^2 + 5x - 7 \\ & x^3 - 3x^2 \qquad\qquad x^3 \div x = x^2 \\ & \quad x^2 + 5x - 7 \\ & \quad x^2 - 3x \qquad\qquad x^2 \div x = x \\ & \quad\quad 8x - 7 \\ & \quad\quad 8x - 24 \qquad\qquad 8x \div x = 8 \\ & \quad\quad\quad 17 \qquad\qquad \text{remainder } 17 \end{array}$$

Answer: $x^2 + x + 8 + 17/(x - 3)$

BY THE REMAINDER THEOREM

- The remainder is $f(3)$ — just evaluate.
- $f(3) = 27 - 2(9) + 5(3) - 7$
- $= 27 - 18 + 15 - 7 = 17$ ✓
- Same answer, no division at all.

And since $17 \neq 0$, $(x - 3)$ is not a factor of $f(x)$.

Both routes always agree. Use the theorem to check your division — or skip the division entirely.

Question R1

2 marks

Remainders and factors

THE QUESTION

Without dividing, find the remainder when $f(x) = x^3 - 2x^2 + 5x - 7$ is divided by $(x - 3)$.
Then state whether $(x - 3)$ is a factor of $f(x)$.

Question R1 — solution

2 marks

Remainders and factors

CONDENSED SOLUTION

- Remainder theorem: the remainder is f(3).
- f(3) = 27 − 2(9) + 5(3) − 7 = 27 − 18 + 15 − 7 = 17.
- Remainder = 17. Since 17 ≠ 0, (x − 3) is NOT a factor of f(x), and x = 3 is not a root.

TRAP Answer the whole question: the number AND the factor statement. “17” on its own drops a mark.

Question R2

3 marks

Remainders and factors

THE QUESTION

$f(x) = x^3 + ax^2 - 5x + 6$. Given that $(x - 1)$ is a factor of $f(x)$, find the value of a.
Then fully factorise $f(x)$.

Question R2 — solution

3 marks

Remainders and factors

CONDENSED SOLUTION

- $(x - 1)$ a factor means $f(1) = 0$: $1 + a - 5 + 6 = 0$, so $a + 2 = 0$ and $a = -2$.
- So $f(x) = x^3 - 2x^2 - 5x + 6$. Divide by $(x - 1)$: coefficients 1, −2, −5, 6.
- Long division: $x^3 \div x = x^2$; subtract → $-x^2$; $-x$; subtract → $-6x$; -6; remainder 0.
- Quotient $x^2 - x - 6 = (x - 3)(x + 2)$.
- $f(x) = (x - 1)(x - 3)(x + 2)$.

TRAP After finding a, rebuild f(x) before dividing — students who skip this step divide the wrong polynomial.

Question R3

3 marks

Remainders and factors

THE QUESTION

$f(x) = 2x^3 - 3x^2 + px + 1$ leaves a remainder of 4 when divided by $(x - 1)$.

(a) Find the value of p.
(b) Write f(x) in the form $(x - 1)\cdot q(x) + r$.

Question R3 — solution

3 marks

Remainders and factors

CONDENSED SOLUTION

- (a) By the remainder theorem the remainder is $f(1) = 2 - 3 + p + 1 = p$. So $p = 4$.
- So $f(x) = 2x^3 - 3x^2 + 4x + 1$.
- (b) Long division by $(x - 1)$: $2x^3 \div x = 2x^2$; subtract → $-x^2$; $-x$; subtract → $3x$; $+3$; subtract → 4.
- $f(x) = (x - 1)(2x^2 - x + 3) + 4$.
- Check: $(x - 1)(2x^2 - x + 3) = 2x^3 - 3x^2 + 4x - 3$, and $+ 4$ gives $2x^3 - 3x^2 + 4x + 1$ ✓

TRAP Set up the equation from the theorem first: remainder = $f(1) = 4$. That single line earns the first mark.

Question R4

3 marks

Remainders and factors

THE QUESTION

Show that $(x + 2)$ is a factor of $f(x) = x^3 + 3x^2 - 4x - 12$, and hence fully factorise $f(x)$.

Question R4 — solution

3 marks

Remainders and factors

CONDENSED SOLUTION

- $(x + 2)$ is a factor means $f(-2) = 0$.
- $f(-2) = -8 + 3(4) - 4(-2) - 12 = -8 + 12 + 8 - 12 = 0$ ✓
- Divide by $(x + 2)$: $x^3 \div x = x^2$; subtract → x^2; $+x$; subtract → $-6x$; -6; remainder 0.
- Quotient $x^2 + x - 6 = (x + 3)(x - 2)$.
- $f(x) = (x + 2)(x + 3)(x - 2)$.

TRAP "Show that" questions need the evaluation written out — $f(-2) = 0$ with the working shown, not just the conclusion.

Mark scheme

Marks by question, with what earns each mark

S1	**5**	1 per part: direction, axis, vertex, intercept, roots
S2	**4**	1 direction+vertex, 1 axis, 1 roots, 1 labelled sketch
A1	**2**	1 dividend/divisor named · 1 identity written correctly
Y1	**3**	1 method (divide–multiply–subtract) · 1 quotient · 1 remainder over divisor
L1	**3**	1 method · 1 quotient · 1 remainder over divisor
L2	**3**	1 method · 1 quotient · 1 remainder, with the check shown
L3	**3**	1 missing terms inserted · 1 method · 1 answer with polynomial remainder
R1	**2**	1 remainder f(3) = 17 · 1 factor statement
R2	**3**	1 finding a · 1 division · 1 full factorisation
R3	**3**	1 finding p · 1 division · 1 identity written correctly
R4	**3**	1 f(−2) = 0 shown · 1 division · 1 full factorisation

Main lesson total: 30 marks. Starter: 9 practice marks (not assessed).

Answer key

One line per question

ANSWERS

- S1 up; x = 2; (2,−2); (0,6); roots 1 and 3
- S2 down; vertex (2,1); x = 2; roots 1 and 3
- A1 $f(x) = (x+2)(x^2+2x-7) + 24$
- Y1 $x^2 + x - 6 + 12/(x+1)$
- L1 $x^2 + 2x - 7 + 24/(x+2)$
- L2 $2x^2 + x + 6 + 7/(x-2)$

QUICK VERIFICATION

- L3 $3x^2 - 8 + (2x+7)/(x^2+1)$
- R1 remainder 17; not a factor
- R2 $a = -2$; $f(x) = (x-1)(x-3)(x+2)$
- R3 $p = 4$; $f(x) = (x-1)(2x^2-x+3) + 4$
- R4 $f(x) = (x+2)(x+3)(x-2)$
- Method check: divisor × quotient + remainder
- must rebuild the dividend exactly.

True or false · 1

T / F

Decide, then justify your decision — answer on the next slide

If the remainder is 0 when f(x) is divided by (x − k), then k is a root of f(x) = 0.

True or false · 1 — answer

TRUE

TRUE

Zero remainder means the divisor divides evenly, so (x − k) is a factor — and by the factor theorem f(k) = 0, so k is a root.

True or false · 2

T / F

Decide, then justify your decision — answer on the next slide

The degree of the quotient equals the degree of the dividend.

True or false · 2 — answer

FALSE

FALSE

deg q = deg f − deg d. Dividing a cubic by a linear divisor gives a quadratic quotient, one degree lower.

True or false · 3

T / F

Decide, then justify your decision — answer on the next slide

A polynomial remainder can be valid, as long as its degree is less than the degree of the divisor.

True or false · 3 — answer

TRUE

TRUE

You stop dividing when deg r < deg d. So r(x) = 2x + 7 with a quadratic divisor is a perfectly valid remainder.

True or false · 4

T / F

Decide, then justify your decision — answer on the next slide

Subtracting in long division means subtracting only the first term of each row.

True or false · 4 — answer

FALSE

FALSE

Subtraction acts on the WHOLE row: every sign flips. (a − b) ≠ (a + b) — this is where most method marks are lost.

True or false · 5

T / F

Decide, then justify your decision — answer on the next slide

Missing terms in the dividend can be skipped as long as you are careful.

True or false · 5 — answer

FALSE

FALSE

Missing terms must be written in with zero coefficients: $x^4 - 16$ becomes $x^4 + 0x^3 + 0x^2 + 0x - 16$. Skipping them shifts every column.

True or false · 6

T / F

Decide, then justify your decision — answer on the next slide

By the remainder theorem, the remainder when f(x) is divided by (x − 3) is f(3).

True or false · 6 — answer

TRUE

TRUE

Exactly. That is why you can find remainders without dividing at all — just evaluate f(k).

True or false · 7

T / F

Decide, then justify your decision — answer on the next slide

If $(x - 2)$ is a factor of $f(x)$, then $f(2) = 0$.

True or false · 7 — answer

TRUE

TRUE

By the factor theorem: a factor $(x - k)$ always satisfies $f(k) = 0$. So $f(2) = 0$ and $x = 2$ is a root.

True or false · 8

T / F

Decide, then justify your decision — answer on the next slide

The remainder when f(x) is divided by (x + 3) is f(3).

True or false · 8 — answer

FALSE

FALSE

$x + 3 = x - (-3)$, so $k = -3$. The remainder is $f(-3)$, not $f(3)$ — the sign flips with the divisor.

FIVE THINGS TO REMEMBER

Do these and algebraic division will not cost you marks

- Write missing terms with zero coefficients before you start.
- The cycle never changes: divide, multiply, subtract, bring down.
- Subtraction flips every sign in the row — check twice.
- The remainder when dividing by (x − k) is f(k) — no division needed.
- Zero remainder means a factor: f(k) = 0 and k is a root.

Built from the 2-04 Dividing Polynomials booklet.