Question 1 (Algebraic manipulation)

Solve:

$$2x^2 - 5x - 3 = 0$$

Question 2 (Hidden quadratic)

Solve:

$$x^4 - 5x^2 + 4 = 0$$

Question 3 (Parameter + roots condition)

Find all values of k such that the equation has **equal roots**:

$$x^2 + kx + 9 = 0$$

Answers

1. $x = 3,\ -\frac{1}{2}$
2. $x = \pm 1,\ \pm 2$
3. $k = \pm 6$

Sum and Roots of Polynomials

Core Vocabulary

- Roots (solutions)（根 / 解）
- Polynomial（多项式）
- Quadratic（二维二次函数）
- Cubic（立方多项式）

Vieta's Relationships

- Sum of roots（根之和）
- Product of roots（根之积）
- Coefficient（系数）
- Leading coefficient（首项系数）

The Roots of Polynomials

Objective: To be able to derive and use the relationships between the roots of a **quadratic** equation

To be able to derive and use the relationships between the roots of a **cubic** equation

To be able to derive and use the relationships between the roots of a **quartic** equation

To be able to evaluate expressions relating to the roots of polynomials

To be able to find the equation of a polynomial whose roots are a **linear transformation** of the roots of a given polynomial

Quadratic Equations

$f(x) = ax^2 + bx + c$

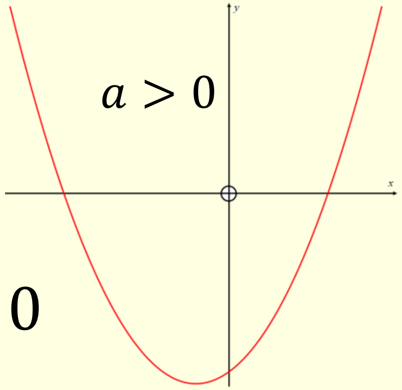
Roots: Solve $f(x) = 0 \quad \Longrightarrow \quad ax^2 + bx + c = 0$

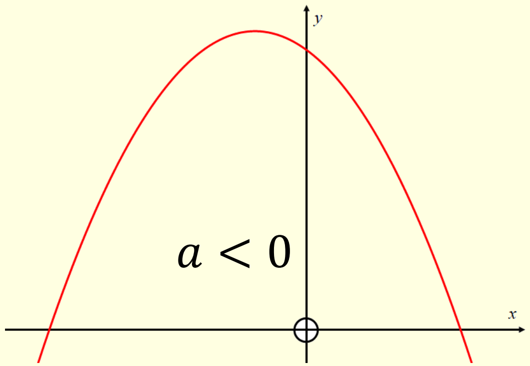
There will be two roots…

- **Two distinct real** roots; α and β

  $f(x)$ has two factors; $(x - \alpha)$ and $(x - \beta)$

  $$\Longrightarrow \quad ax^2 + bx + c = a(x - \alpha)(x - \beta) = 0$$

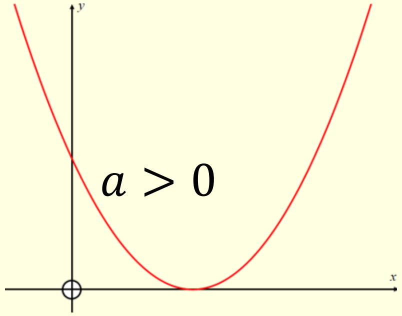


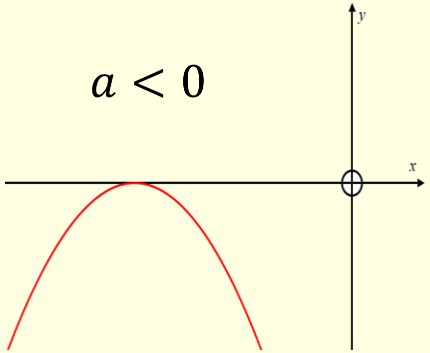


- **One repeated real** root; α

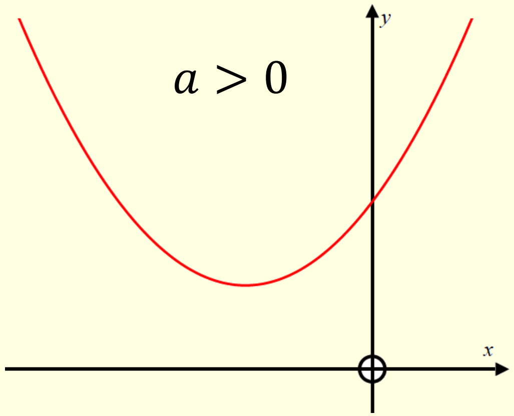
  $f(x)$ has a repeated factor; $(x - \alpha)$

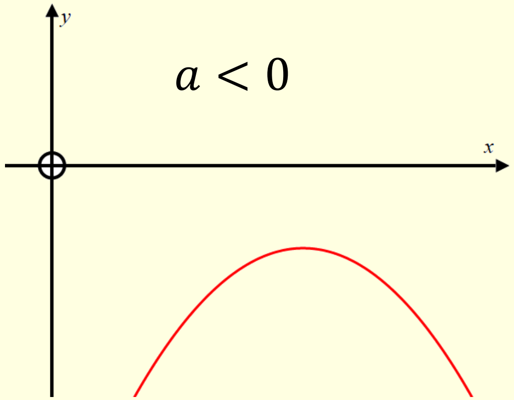
  $$\Longrightarrow \quad ax^2 + bx + c = a(x - \alpha)^2 = 0$$





- A **pair of complex conjugate roots**; $\alpha = p + iq$ and $\beta = p - iq$





$f(x)$ has a factors $[x-(p+iq)]$ *and* $[x-(p-iq)]$

$\Longrightarrow \quad ax^2 + bx + c = a[x-(p+iq)][x-(p-iq)] = 0$

$a, b, c \in \mathbb{R}$
$x \in \mathbb{C}$

Two distinct real roots; α and β

$f(x)$ has two factors; $(x-\alpha)$ and $(x-\beta)$

$$ax^2+bx+c \equiv a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]$$

$$= a[(x-\alpha)(x-\beta)]$$

$$= a[x^2-\beta x-\alpha x+\alpha\beta]$$

$$= a[x^2-(\alpha+\beta)x+\alpha\beta]$$

$$= a\left[x^2-\left(\sum\alpha\right)x+\left(\prod\alpha\right)\right]$$

$\sum$ Sigma → "The **S**um of "

$\prod$ Pi → "The **P**roduct of "

$\therefore \quad {}^{-}(\alpha+\beta) = \dfrac{b}{a}$

$\Rightarrow \quad \alpha+\beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$ Standard results that can be quoted in exams.

Example: $f(x) = ax^2 + bx + c; \quad a, b, and\ c \in \mathbb{Z},$ has roots $\alpha = \frac{3}{4}$ and $\beta = -\frac{2}{5}$

Determine the values of a, b and c

$$\frac{b}{a} = {}^{-}(\alpha + \beta) = {}^{-}\left(\frac{3}{4} + {}^{-}\frac{2}{5}\right) = {}^{-}\left(\frac{15}{20} - \frac{8}{20}\right) = {}^{-}\frac{7}{20}$$

$$\frac{c}{a} = \alpha\beta = \frac{3}{4} \times {}^{-}\frac{2}{5} = {}^{-}\frac{6}{20}$$

$$f(x) = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = 0 \implies 20\left[x^2 + {}^{-}\frac{7}{20}x + {}^{-}\frac{6}{20}\right] = 0$$

$$20x^2 - 7x - 6 = 0$$

$\therefore \quad a = 20, \quad b = {}^{-}7, \quad c = {}^{-}6$

Alternatively…

$$ax^2 + bx + c \equiv a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = 0$$

$$\div a \quad a\left[\left(x - \frac{3}{4}\right)\left(x - \frac{{}^{-}2}{5}\right)\right] = 0 \quad \div a$$

$$\times 4 \quad \times 5 \quad \times 4 \ \times 5$$

$$\left(x - \frac{3}{4}\right)\left(x + \frac{2}{5}\right) = 0$$

$$(4x - 3)(5x + 2) = 0$$

$$20x^2 + 8x - 15x - 6 = 0$$

$$20x^2 - 7x - 6 = 0$$

Example: $f(x) = 3x^2 - x - 4$ has roots α and β

Without solving the equation, find the values of:

i. $\alpha + \beta$ *ii.* $\alpha\beta$ *iii.* $\frac{1}{\alpha} + \frac{1}{\beta}$ *iv.* $\alpha^2 + \beta^2$

i. & ii. $f(x) = 3x^2 - x - 4$

$$= 3\left[x^2 - \frac{1}{3}x - \frac{4}{3}\right] \Rightarrow \alpha + \beta = -\left(-\frac{1}{3}\right) \Rightarrow \alpha + \beta = \frac{1}{3} \quad \& \quad \alpha\beta = -\frac{4}{3}$$

iii.

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta}{\alpha\beta} + \frac{\alpha}{\alpha\beta}$$

$$= \frac{\alpha + \beta}{\alpha\beta}$$

$$= \frac{\frac{1}{3}}{-\frac{4}{3}}$$

$$= -\frac{1}{4}$$

iv.

$$\alpha^2 + \beta^2 = \alpha^2 + \beta^2(+2\alpha\beta - 2\alpha\beta)$$

$$= (\alpha^2 + 2\alpha\beta + \beta^2) - 2\alpha\beta$$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \left(\frac{1}{3}\right)^2 - 2 \times -\frac{4}{3}$$

$$= \frac{1}{9} + \frac{8}{3}$$

$$= \frac{1}{9} + \frac{24}{9}$$

$$= \frac{25}{9}$$

Cubic Equations

$f(x) = ax^3 + bx^2 + cx + d$

Roots: Solve $f(x) = 0 \quad \Longrightarrow \quad ax^3 + bx^2 + cx + d = 0$

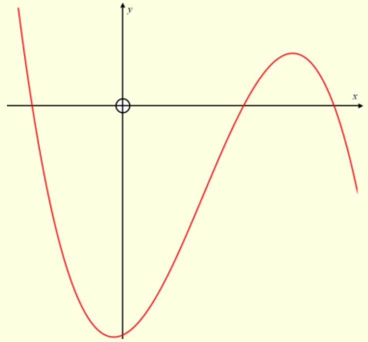
There will be three roots…

- Three distinct real roots; α, β *and* γ

  $f(x)$ has factors $(x-\alpha), (x-\beta)$ and $(x-\gamma)$

  $\Longrightarrow \quad ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma) = 0$

$a < 0$

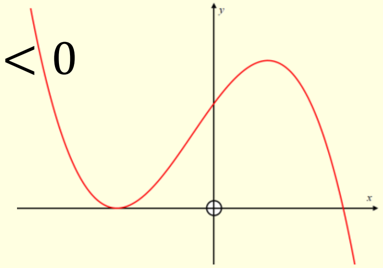


- A repeated real root, α, and a second real root β

  $f(x)$ has factors $(x-\alpha)$,$(x-\alpha)$ and $(x-\beta)$

  $\Longrightarrow \quad ax^3 + bx^2 + cx + d = a(x-\alpha)^2(x-\beta) = 0$

$a < 0$

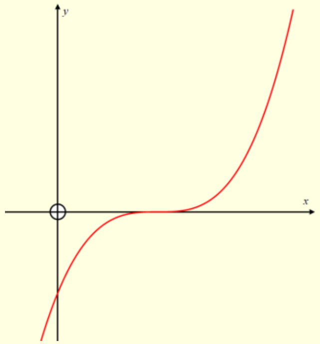


- One repeated real root; α

  $f(x)$ has factors $(x-\alpha)$,$(x-\alpha)$ and $(x-\alpha)$

  $\Longrightarrow \quad ax^3 + bx^2 + cx + d = \; a(x-\alpha)^3 = 0$

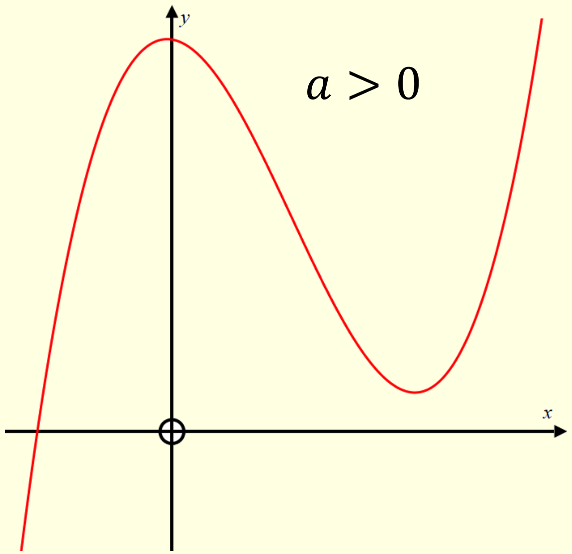
$a > 0$



- One real root, α, and a complex conjugate complex pair, $\beta = p + iq$ and $\gamma = p - iq$

$f(x)$ has factors $(x - \alpha)$,$[x - (p + iq)]$, and $[x - (p - iq)]$

$$\Longrightarrow \quad ax^3 + bx^2 + cx + d = a(x - \alpha)[x - (p + iq)][x - (p - iq)] = 0$$



Three distinct real roots; α, β and γ

$f(x)$ has three factors; $(x-\alpha), (x-\beta)$ and $(x-\gamma)$

$$ax^3+bx^2+cx+d = a\left[x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}\right]$$

$$= a[(x-\alpha)(x-\beta)(x-\gamma)]$$

$$= a[(x^2-\beta x-\alpha x+\alpha\beta)(x-\gamma)]$$

$$= a[(x^2-(\alpha+\beta)x+\alpha\beta)(x-\gamma)]$$

$$= a[x^3-(\alpha+\beta)x^2+\alpha\beta x-\gamma x^2+\gamma(\alpha+\beta)x-\alpha\beta\gamma]$$

$$= a[x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\alpha\gamma+\beta\gamma)x-\alpha\beta\gamma]$$

$$\therefore \quad ax^3+bx^2+cx+d = a\left[x^3-\left(\sum\alpha\right)x^2+\left(\sum\alpha\beta\right)x-\left(\prod\alpha\right)\right]$$

$^{-}(\alpha+\beta+\gamma) = \frac{b}{a} \implies \alpha+\beta+\gamma = {}^{-}\frac{b}{a}$ $\quad \alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$ $\quad {}^{-}\alpha\beta\gamma = \frac{d}{a} \implies \alpha\beta\gamma = {}^{-}\frac{d}{a}$

Example: $f(x) = 2x^3 + 5x^2 - 3x - 7$ has roots $\alpha,\ \beta$ and γ

Without solving the equation, find the values of:

i. $\alpha + \beta + \gamma$ *ii.* $\alpha\beta + \alpha\gamma + \beta\gamma$ *iii.* $\alpha\beta\gamma$ *iv.* $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ *v.* $\alpha^2 + \beta^2 + \gamma^2$

i., ii. and iii. $f(x) = 2x^3 + 5x^2 - 3x - 7$

$$= 2\left(x^3 + \frac{5}{2}x^2 + \frac{^-3}{2}x + \frac{^-7}{2}\right)$$

$\therefore$ $\alpha + \beta + \gamma = -\frac{5}{2}$ $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{^-3}{2}$ $\alpha\beta\gamma = -\frac{^-7}{2} \Longrightarrow \alpha\beta\gamma = \frac{7}{2}$

iv. $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$$= \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$$

Quotable Result

$$= \frac{\frac{^-3}{2}}{\frac{7}{2}}$$

$$= -\frac{3}{7}$$

v. $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - [\alpha(\beta + \gamma) + \beta(\alpha + \gamma) + \gamma(\alpha + \beta)]$

$$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

Quotable Result

$$= \left(-\frac{5}{2}\right)^2 - 2\left(\frac{^-3}{2}\right)$$

$$= \frac{25}{4} + 3$$

$$= \frac{37}{4}$$

Example: Find a cubic equation with integer coefficients which has roots $2+3i$ and 5 (and a third root that is to be determined)

If $2+3i$ is a root, then so is $2-3i$

Let $\alpha = 2+3i$, $\beta = 2-3i$ and $\gamma = 5$

$$\alpha+\beta+\gamma = (2+3i)+(2-3i)+5 = 9$$

$$\alpha\beta\gamma = 5(2+3i)(2-3i) = 5(4-6i+6i+9) = 5\times 13 = 65$$

$$\alpha\beta+\alpha\gamma+\beta\gamma = (2+3i)(2-3i)+5(2+3i)+5(2-3i)$$
$$= 4-6i+6i+9+10+15i+10-15i$$
$$= 4-6i+6i+9+10+15i+10-15i$$
$$= 33$$

$$a[x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\alpha\gamma+\beta\gamma)x-\alpha\beta\gamma] = 0$$

$$a(x^3-9x^2+33x-65) = 0$$

Let $a = 1$ $\quad\Longrightarrow\quad$ $x^3-9x^2+33x-65 = 0$

Alternative Solution

Example: Find a cubic equation with integer coefficients which has roots $2+3i$ and 5 (and a third root that is to be determined)

If $2+3i$ is a root, then so is $2-3i$

$$f(x) = [x-(2+3i)][x-(2-3i)](x-5)$$
$$= [x^2-(2-3i)x-(2+3i)x+(2+3i)(2-3i)](x-5)$$
$$= [x^2-4x+4-6i+6i+9](x-5)$$
$$= (x^2-4x+13)(x-5)$$
$$= x^3-4x^2+13x-5x^2+20x-65$$
$$= x^3-9x^2+33x-65$$

$\therefore$ $x^3-9x^2+33x-65=0$ is a cubic equation with integer coefficients which has roots $2+3i, 2-3i$ and 5

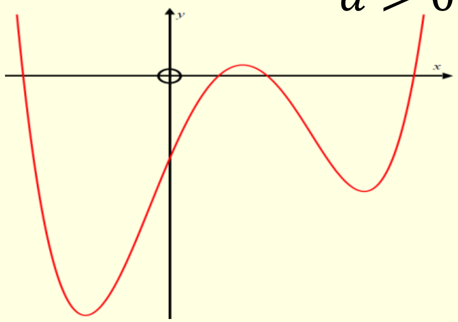
Quartic Equations

$f(x) = ax^4 + b3 + cx^2x + dx + e$

Roots: Solve $f(x) = 0 \quad \Longrightarrow \quad ax^4 + bx^3 + cx^2x + dx + e = 0$

There will be four roots…

$a > 0$

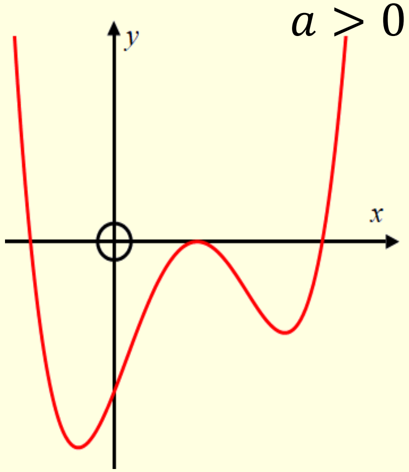


- Four distinct real roots; α, β, γ and δ

  $f(x)$ has factors $(x - \alpha), (x - \beta), (x - \gamma)$ and $(x - \delta)$

  $\Longrightarrow \quad ax^4 + bx^3 + cx^2x + dx + e = a(x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$

$a > 0$



- A real root, α, repeated twice, and two other real roots; $\beta \; and \; \gamma$

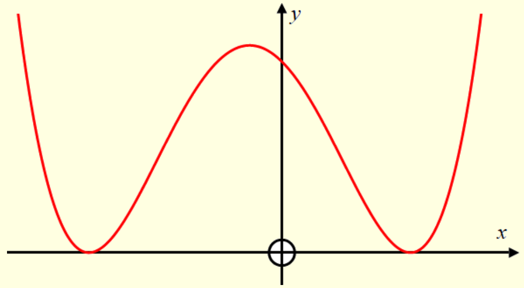
  $f(x)$ has factors $(x - \alpha)$,$(x - \alpha), (x - \beta)$and $(x - \gamma)$

  $\Longrightarrow \quad ax^4 + bx^3 + cx^2x + dx + e = a(x - \alpha)^2(x - \beta)(x - \gamma)$

- Two distinct real roots, α $and\ \beta$, each repeated twice

  $f(x)$ has factors $(x-\alpha),(x-\alpha),(x-\beta)$and $(x-\beta)$

  $\Longrightarrow\ ax^4+bx^3+cx^2x+dx+e=\ a(x-\alpha)^2(x-\beta)^2=0$

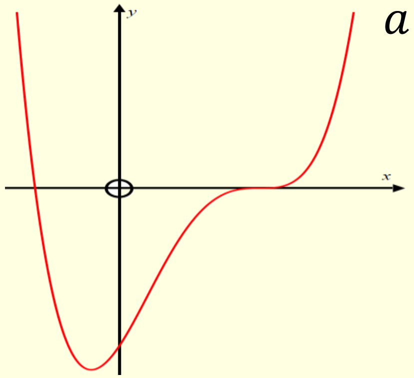


$a>0$

- A real root, α, repeated three times, and another real root, β

  $f(x)$ has factors $(x-\alpha),(x-\alpha),(x-\alpha)$and $(x-\beta)$

  $\Longrightarrow\ ax^4+bx^3+cx^2x+dx+e=\ a(x-\alpha)^3(x-\beta)=0$

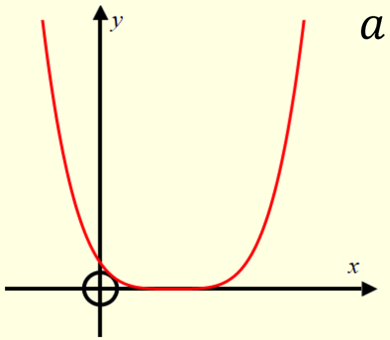


$a>0$

- A real root, α, repeated four times.

  $f(x)$ has factors $(x-\alpha),(x-\alpha),(x-\alpha)$ and $(x-\alpha)$

  $\Longrightarrow\ ax^4+bx^3+cx^2x+dx+e=\ a(x-\alpha)^4=0$

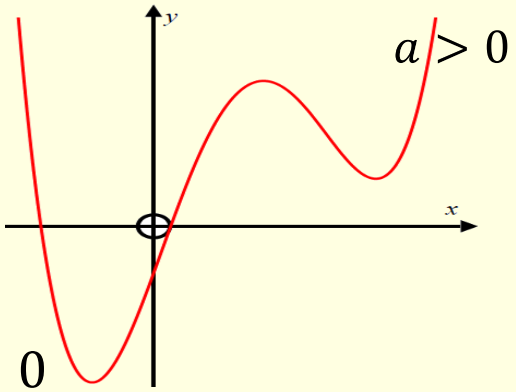


$a>0$

- Two distinct real roots, α and β, and a **pair of complex conjugate roots**; $\gamma = p + iq$ and $\delta = p - iq$

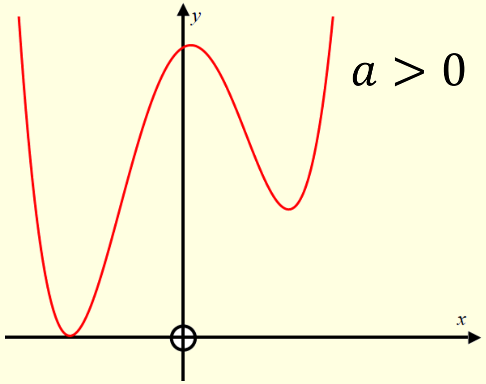
  $f(x)$ has factors $(x - \alpha)$, $(x - \beta)$, $[x - (p + iq)], [x - (p - iq)]$

  $ax^4 + bx^3 + cx^2x + dx + e = a(x - \alpha)(x - \beta)[x - (p + iq)][x - (p - iq)] = 0$



- A real root, α, repeated twice, and a **pair of complex conjugate roots**; $\beta = p + iq$ and $\gamma = pa - iq$

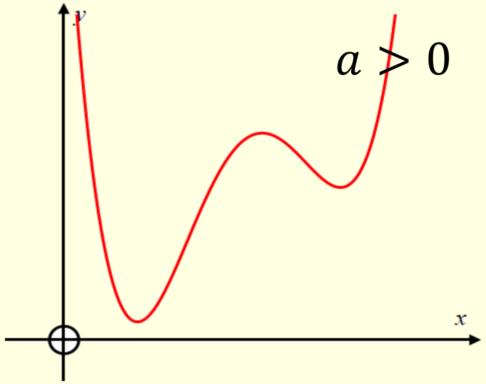
  $ax^4 + bx^3 + cx^2x + dx + e = a(x - \alpha)^2[x - (p + iq)][x - (p - iq)] = 0$



- Two **pairs of complex conjugate roots**;

  $\alpha = p + iq, \beta = p - iq, \gamma = r + is \; and \; \delta = r - is$

  $f(x) = a[x - (p + iq)][x - (p - iq)][x - (r + is)][x - (r - is)] = 0$



Four distinct real roots; α, β, γ and δ

$f(x) = ax^4 + bx^3 + cx^2 + dx + e$ has four factors $(x-\alpha), (x-\beta), (x-\gamma)$ and $(x-\delta)$

$$\Rightarrow \; f(x) = a\left[x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a}\right]$$

$$= a[(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)]$$

$$= a[(x^3 - (\alpha+\beta)x^2 + \alpha\beta x - \gamma x^2 + \gamma(\alpha+\beta)x - \alpha\beta\gamma)(x-\delta)]$$

$$= a\begin{bmatrix} x^4 - (\alpha+\beta)x^3 + \alpha\beta x^2 - \gamma x^3 + \gamma(\alpha+\beta)x^2 - \alpha\beta\gamma x \\ -\delta x^3 + (\alpha+\beta)\delta x^2 - \alpha\beta\delta x + \gamma\delta x^2 - \gamma\delta(\alpha+\beta)x + \alpha\beta\gamma\delta \end{bmatrix}$$

$$= a\begin{bmatrix} x^4 - (\alpha+\beta+\gamma+\delta)x^3 + (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x^2 \\ -(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)x + \alpha\beta\gamma\delta \end{bmatrix}$$

$$\therefore \; f(x) = a\left[x^4 - \left(\sum \alpha\right)x^3 + \left(\sum \alpha\beta\right)x^2 - \left(\sum \alpha\beta\gamma\right)x + \left(\prod \alpha\right)\right]$$

$\alpha+\beta+\gamma+\delta = -\frac{b}{a}$ $\quad$ $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a}$ $\quad$ $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -\frac{d}{a}$ $\quad$ $\alpha\beta\gamma\delta = \frac{e}{a}$

$\sum \alpha = -\frac{b}{a}$ $\quad$ $\sum \alpha\beta = \frac{c}{a}$ $\quad$ $\sum \alpha\beta\gamma = -\frac{d}{a}$ $\quad$ $\prod \alpha = \frac{e}{a}$

Example: The equation $x^4 + 2x^3 + px^2 + qx - 60 = 0, x \in \mathbb{C}, p \,\&\, q \in \mathbb{R}$, has roots $\alpha, \beta, \gamma, \delta$.
Given that $\gamma = {}^-2 + 4i$ and $\delta = \gamma^*$

a) Show that $\alpha + \beta - 2 = 0$ and that $\alpha\beta + 3 = 0$

b) Hence find all the roots of the quartic equation and find the values of p and q

a) $\delta = \gamma^* \quad \Longrightarrow \quad \delta = {}^-2 - 4i$

$$\sum \alpha = {}^-\frac{b}{a} \quad \Longrightarrow \quad \alpha + \beta + \gamma + \delta = {}^-\frac{2}{1}$$

$$\alpha + \beta + {}^-2 + 4i + {}^-2 - 4i = {}^-2$$

$$\alpha + \beta - 2 = 0 \quad QED$$

$$\alpha\beta\gamma\delta = \frac{d}{a} \quad \Longrightarrow \quad \alpha\beta({}^-2 + 4i)({}^-2 - 4i) = {}^-60$$

$$\alpha\beta(4 + 16) = {}^-60$$

$$\alpha\beta = {}^-3$$

$$\alpha\beta + 3 = 0 \quad QED$$

b) Hence find all the roots of the quartic equation and find the values of p and q

$\alpha + \beta - 2 = 0 \implies \alpha^2 + \alpha\beta - 2\alpha = 0$

$\alpha\beta + 3 = 0 \implies \alpha\beta = {}^{-}3$

$\implies \alpha^2 + {}^{-}3 - 2\alpha = 0$

$\alpha^2 - 2\alpha - 3 = 0$

$(\alpha - 3)(\alpha + 1) = 0$

$\therefore \quad \alpha = {}^{-}1, \alpha = 3$

$\alpha = {}^{-}1 \implies {}^{-}1 + \beta - 2 = 0$

$\beta = 3$

$\alpha = 3 \implies 3 + \beta - 2 = 0$

$\beta = {}^{-}1$

$x^4 + 2x^3 + px^2 + qx - 60 = 0$

$\alpha = {}^{-}1 \qquad \gamma = {}^{-}2 + 4i$

$\beta = 3 \qquad \delta = {}^{-}2 - 4i$

$\implies \sum \alpha\beta = \frac{p}{1}$

$p = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

$= {}^{-}3 + (2 - 4i) + (2 + 4i) + 3({}^{-}2 + 4i)$
$+3({}^{-}2 - 4i) + ({}^{-}2 + 4i)({}^{-}2 - 4i)$

$= {}^{-}3 + 4 - 12 + 20$

$= 9$

$\implies \sum \alpha\beta\gamma = -\frac{q}{1}$

$q = {}^{-}(\alpha\beta\gamma + \alpha\beta\delta + \alpha\delta\gamma + \beta\gamma\delta)$

$= {}^{-}(6 - 12i + 6 + 12i - 20 + 60)$

$= {}^{-}52$

Useful Algebraic Forms

Reciprocals

$$\frac{1}{\alpha}+\frac{1}{\beta} \equiv \frac{\alpha+\beta}{\alpha\beta} \equiv \frac{\sum \alpha}{\prod \alpha}$$

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} \equiv \frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma} \equiv \frac{\sum \alpha\beta}{\prod \alpha}$$

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta} \equiv \frac{\beta\gamma\delta+\alpha\gamma\delta+\alpha\beta\delta+\alpha\beta\gamma}{\alpha\beta\gamma\delta} \equiv \frac{\sum \alpha\beta\gamma}{\prod \alpha}$$

Products of Powers

$$\alpha^n \times \beta^n \equiv (\alpha\beta)^n$$

$$\alpha^n \times \beta^n \times \gamma^n \equiv (\alpha\beta\gamma)^n$$

$$\alpha^n \times \beta^n \times \gamma^n \times \delta^n \equiv (\alpha\beta\gamma\delta)^n$$

The Sums of Squares

$$(\alpha + \beta)^2 \equiv \alpha^2 + 2\alpha\beta + \beta^2$$

$$\Rightarrow \qquad \alpha^2 + \beta^2 \equiv (\alpha + \beta)^2 - 2\alpha\beta$$

$$\sum \alpha^2 \equiv \left(\sum \alpha\right)^2 - 2\sum \alpha\,\beta$$

$$(\alpha + \beta + \gamma)^2 \equiv \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$\Rightarrow \qquad \alpha^2 + \beta^2 + \gamma^2 \equiv (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$\sum \alpha^2 \equiv \left(\sum \alpha\right)^2 - 2\sum \alpha\,\beta$$

$$\Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \equiv (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

$$\sum \alpha^2 \equiv \left(\sum \alpha\right)^2 - 2\sum \alpha\,\beta$$

The Sums of Cubes

$$(\alpha+\beta)^3 \equiv \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$$

$$\Longrightarrow \quad \alpha^3+\beta^3 \equiv (\alpha+\beta)^3 - (3\alpha^2\beta + 3\alpha\beta^2)$$

$$\boldsymbol{\alpha^3+\beta^3 \equiv (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}$$

$$\sum\alpha^3 \equiv \left(\sum\alpha\right)^3 - 3\prod\alpha\sum\alpha$$

$$(\alpha+\beta+\gamma)^3 \equiv \alpha^3 + 3\alpha^2(\beta+\gamma) + 3\alpha(\beta+\gamma)^2 + (\beta+\gamma)^3$$

$$\Longrightarrow \quad \equiv \alpha^3 + 3\alpha(\beta+\gamma)(\alpha+\beta+\gamma) + \beta^3 + 3\beta^2\gamma + 3\beta\gamma^2 + \gamma^3$$

$$\equiv \alpha^3 + \beta^3 + \gamma^3 + 3\alpha(\beta+\gamma)(\alpha+\beta+\gamma) + 3\beta\gamma(\beta+\gamma)$$

$$\equiv \alpha^3 + \beta^3 + \gamma^3 + 3(\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma) + 3\alpha^2(\beta+\gamma)$$

$$\equiv \alpha^3 + \beta^3 + \gamma^3 + 3(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma) + 3\alpha^2(\beta+\gamma) - 3\alpha(\alpha\beta+\alpha\gamma+\beta\gamma)$$

$$\equiv \alpha^3 + \beta^3 + \gamma^3 + 3(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma) + 3\alpha^2(\beta+\gamma) - 3\alpha^2(\beta+\gamma) - 3\alpha\beta\gamma$$

$$\equiv \alpha^3 + \beta^3 + \gamma^3 + 3(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma) - 3\alpha\beta\gamma$$

$$\Longrightarrow \quad \boldsymbol{\alpha^3+\beta^3+\gamma^3 \equiv (\alpha+\beta+\gamma)^3 - 3(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma) + 3\alpha\beta\gamma}$$

$$\sum\alpha^3 \equiv \left(\sum\alpha\right)^3 - 3\sum\alpha\sum\alpha\beta + 3\prod\alpha$$

Linear Transformations of Roots

Given the **sums** and **products** of the roots of a polynomial, it is possible to find the equation of a second polynomial whose roots are a linear transformation of the roots of the first.

For example, if the roots of a cubic equation are α, β *and* γ, we can find the equation of a polynomial with roots $(\alpha+2), (\beta+2)$ and $(\beta+2)$, or $3\alpha, 3\beta$ and 3γ.

Example: The cubic equation $x^3 - 4x^2 + x + 6 = 0$ has roots α, β and γ.
Find the equations of the polynomials with roots:

i. $2\alpha, 2\beta$ and 2γ

ii. $\alpha - 2, \beta - 2, \gamma - 2$

i. $f(x) = a\left[x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}\right] = 0 \implies \left[x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}\right] = x^3 + \frac{{}^-4}{1}x^2 + \frac{1}{1}x + \frac{6}{1} = 0$

$\alpha + \beta + \gamma = {}^-\frac{{}^-4}{1} = 4$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{1}{1} = 1$

$\alpha\beta\gamma = {}^-\frac{6}{1} = {}^-6$

$2\alpha + 2\beta + 2\gamma = 2(\alpha + \beta + \gamma)$
$= 8$

$2\alpha \times 2\beta + 2\alpha \times 2\gamma + 2\beta \times 2\gamma = 4(\alpha\beta + \alpha\gamma + \beta\gamma)$
$= 4 \times 1$
$= 4$

$2\alpha \times 2\beta \times 2\gamma = 8\alpha\beta\gamma$
$= 8 \times {}^-6$
$= {}^-48$

$\therefore$ $x^3 - 8x^2 + 4x + 48 = 0$ has roots $2\alpha, 2\beta$ and 2γ

$x^3 - 4x^2 + x + 6 = 0$ has roots α, β and γ

ii. For roots $\alpha - 2, \beta - 2,$ and $\gamma - 2$

$\alpha + \beta + \gamma = 4 \qquad \alpha\beta + \alpha\gamma + \beta\gamma = 1 \qquad \alpha\beta\gamma = {}^{-}6$

$$(\alpha - 2) + (\beta - 2) + (\gamma - 2) = (\alpha + \beta + \gamma) - 6$$
$$= 4 - 6$$
$$= {}^{-}2$$

$$(\alpha - 2)(\beta - 2) + (\alpha - 2)(\gamma - 2) + (\beta - 2)(\gamma - 2) = \alpha\beta - 2\alpha - 2\beta + 4 + \alpha\gamma - 2\alpha - 2\gamma + 4 + \beta\gamma - 2\beta - 2\gamma + 4$$
$$= \alpha\beta + \alpha\gamma + \beta\gamma - 4(\alpha + \beta + \gamma) + 12$$
$$= 1 - 4 \times 4 + 12$$
$$= {}^{-}3$$

$$(\alpha - 2)(\beta - 2)(\gamma - 2) = (\alpha\beta - 2\alpha - 2\beta + 4)(\gamma - 2)$$
$$= \alpha\beta\gamma - 2\alpha\gamma - 2\beta\gamma + 4\gamma - 2\alpha\beta + 4\alpha + 4\beta - 8$$
$$= \alpha\beta\gamma - 2(\alpha\gamma + \beta\gamma + \alpha\beta) + 4(\gamma + \alpha + \beta) - 8$$
$$= {}^{-}6 - 2 \times 1 + 4 \times 4 - 8$$
$$= 0$$

$\therefore \; x^3 - {}^{-}2x^2 + {}^{-}3x = 0 \implies x^3 + 2x^2 - 3x = 0$ has roots $\alpha - 2, \beta - 2,$ and $\gamma - 2$

Alternative Solution

Example: The cubic equation $x^3 - 4x^2 + x + 6 = 0$ has roots α, β and γ.
Find the equations of the polynomials with roots:

i. $2\alpha, 2\beta$ and 2γ *ii.* $\alpha - 2, \beta - 2, \gamma - 2$

i. The roots of the new equation are double the roots of the old equation.

This is a stretch, scale factor 2, parallel to the *x*-axis $\Longrightarrow$ $f(x) \longrightarrow f\left(\frac{x}{2}\right)$

$$\therefore \quad x^3 - 4x^2 + x + 6 = 0 \quad \longrightarrow \quad \left(\frac{x}{2}\right)^3 - 4\left(\frac{x}{2}\right)^2 + \frac{x}{2} + 6 = 0$$

$$\frac{x^3}{8} - 4 \times \frac{x^2}{4} + \frac{x}{2} + 6 = 0$$

$$x^3 - 8x^2 + 4x + 48 = 0 \text{ has roots } 2\alpha, 2\beta \text{ and } 2\gamma$$

i. The roots of the new equation are two less than the roots of the old equation.

This is a translation, two units **left**, parallel to the *x*-axis $\Longrightarrow$ $f(x) \longrightarrow f(x+2)$

$$\therefore \quad x^3 - 4x^2 + x + 6 = 0 \quad \longrightarrow \quad (x+2)^3 - 4(x+2)^2 + (x+2) + 6 = 0$$

$$\Longrightarrow \quad x^3 + 6x^2 + 12x + 8 - 4(x^2 + 4x + 4) + x + 8 = 0$$

$$x^3 + 2x^2 - 3x = 0$$

$$\therefore \quad x^3 + 2x^2 - 3x = 0 \text{ has roots } \alpha - 2, \beta - 2, \text{and } \gamma - 2$$