Midpoint of a Line Segment Using the Formula

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Quadratics (Challenging A-level)

Question:
Let $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

Given that:

- The graph of $f(x)$ intersects the x-axis at two distinct points,
- $f(1) = 2$,
- The minimum value of $f(x)$ is -3,

(a) Show that $b^2 - 4ac > 0$.
(b) Find a, b, c.
(c) Solve the inequality $f(x) \leq 0$.

Functions (Challenging A-level)

Question:
Let $f(x) = \ln(x - 1)$ and $g(x) = \frac{3}{x}$, where $x > 1$.

(a) Find an expression for $(f \circ g)(x)$ and state its domain.
(b) Solve the equation

$$(f \circ g)(x) = 0.$$

(c) Hence, or otherwise, solve

$$\ln\left(\frac{3}{x} - 1\right) \geq 0.$$

(d) Show that the equation $\ln(x - 1) = \frac{3}{x}$ has exactly one solution, and justify your answer.

Vertex form:

$$f(x) = a(x-h)^2 - 3$$

$f(1) = 2 \Rightarrow a(1-h)^2 = 5.$
Take $a = 1 \Rightarrow h = 1 \pm \sqrt{5}.$

$$b = -2h, \quad c = h^2 - 3$$

One solution:

$$a = 1,\ b = -2 + 2\sqrt{5},\ c = 3 - 2\sqrt{5}$$

Solve $f(x) \leq 0$:

$$(x-h)^2 \leq 3 \Rightarrow x \in [h - \sqrt{3},\ h + \sqrt{3}]$$

Functions

$$(f \circ g)(x) = \ln\left(\frac{3}{x} - 1\right), \quad 0 < x < 3$$

$$\ln\left(\frac{3}{x} - 1\right) = 0 \Rightarrow x = \tfrac{3}{2}$$

$$\ln\left(\frac{3}{x} - 1\right) \geq 0 \Rightarrow 0 < x \leq \tfrac{3}{2}$$

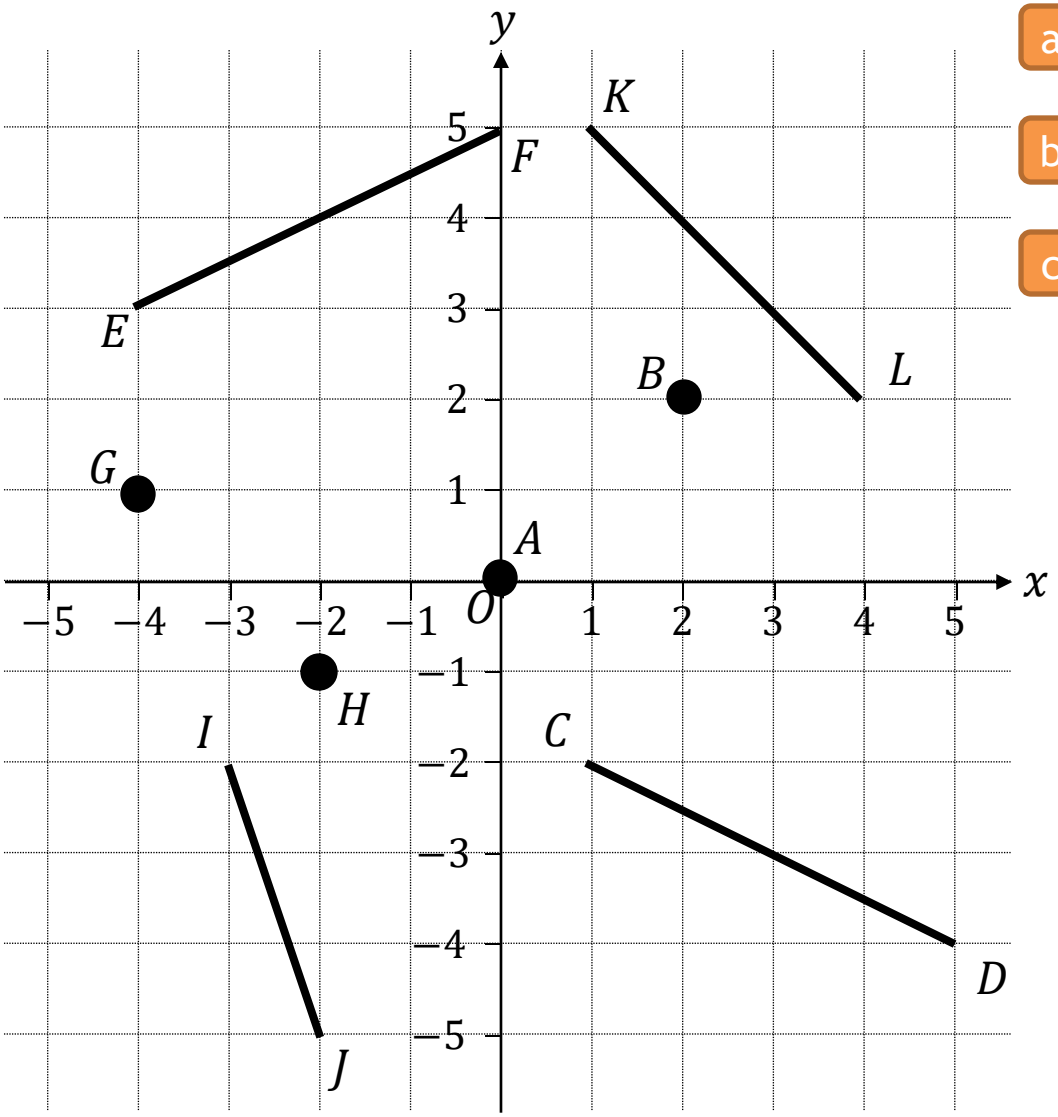
$$\ln(x-1) = \frac{3}{x} : \text{ increasing} = \text{decreasing} \Rightarrow 1 \text{ solution}$$

- Midpoint — 中点
- Line segment — 线段
- Coordinate — 坐标
- Endpoint — 端点
- Average — 平均值
- Mean value — 平均值（或均值）
- Bisection — 平分
- Bisect — 平分 / 二等分
- Distance — 距离
- Horizontal displacement — 水平位移
- Vertical displacement — 垂直位移
- Cartesian plane — 直角坐标平面
- Ordered pair — 有序数对

RECAP: Midpoint of Numbers & by Inspection

Show all solutions

1 Find the midpoint of the following coordinates and line segments:



a M_{AB}: ?

b M_{CD}: ?

c M_{EF}: ?

d M_{GH}: ?

e M_{IJ}: ?

f M_{KL}: ?

2 Find the midpoint of the following numbers:

a 3 and 9 ?

b 5 and 32 ?

c 102 and 70 ?

d 15.65 and 16.3 ?

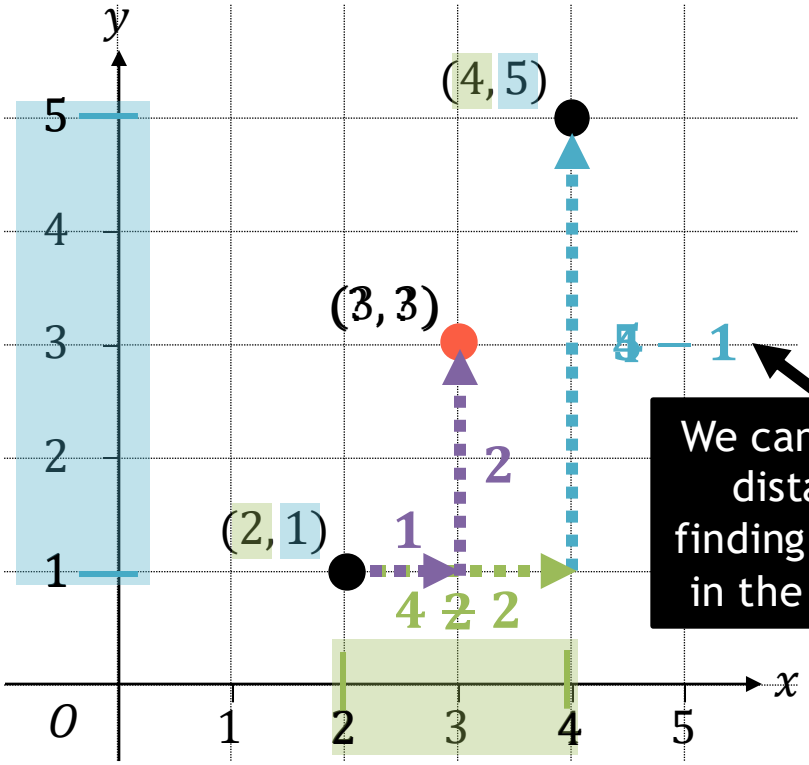
e -20 and 5 ?

f -170 and -128 ?

g $\frac{1}{3}$ and $\frac{4}{5}$?

The Big Idea: Midpoint Without Coordinate Grids

We can still use the horizontal and vertical distances to find midpoint…



We can find the total distance in y by finding the difference in the y-coordinates

We can find the total distance in x by finding the difference in the x-coordinates

Even when there isn't a coordinate grid, or labelled axes!

As the midpoint is exactly halfway between the two endpoints, we need to find the distance in the x- & y-direction.

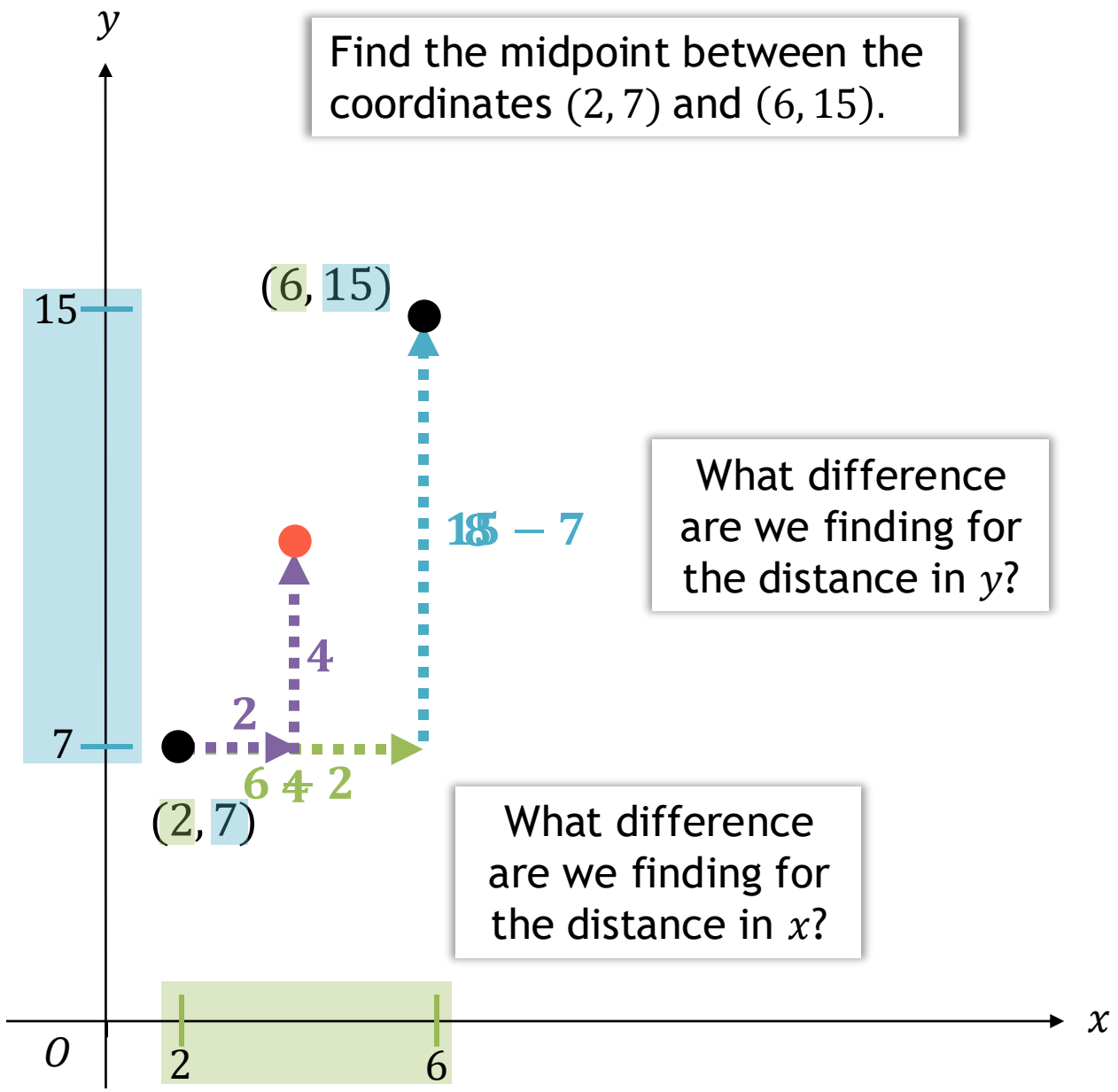
Halve each distance.

Then add these distances to the initial x-and y-coordinates.

$$(2+1, 1+2)$$

$$(3, 3)$$

Example



Find the midpoint between the coordinates $(2, 7)$ and $(6, 15)$.

What difference are we finding for the distance in y?

What difference are we finding for the distance in x?

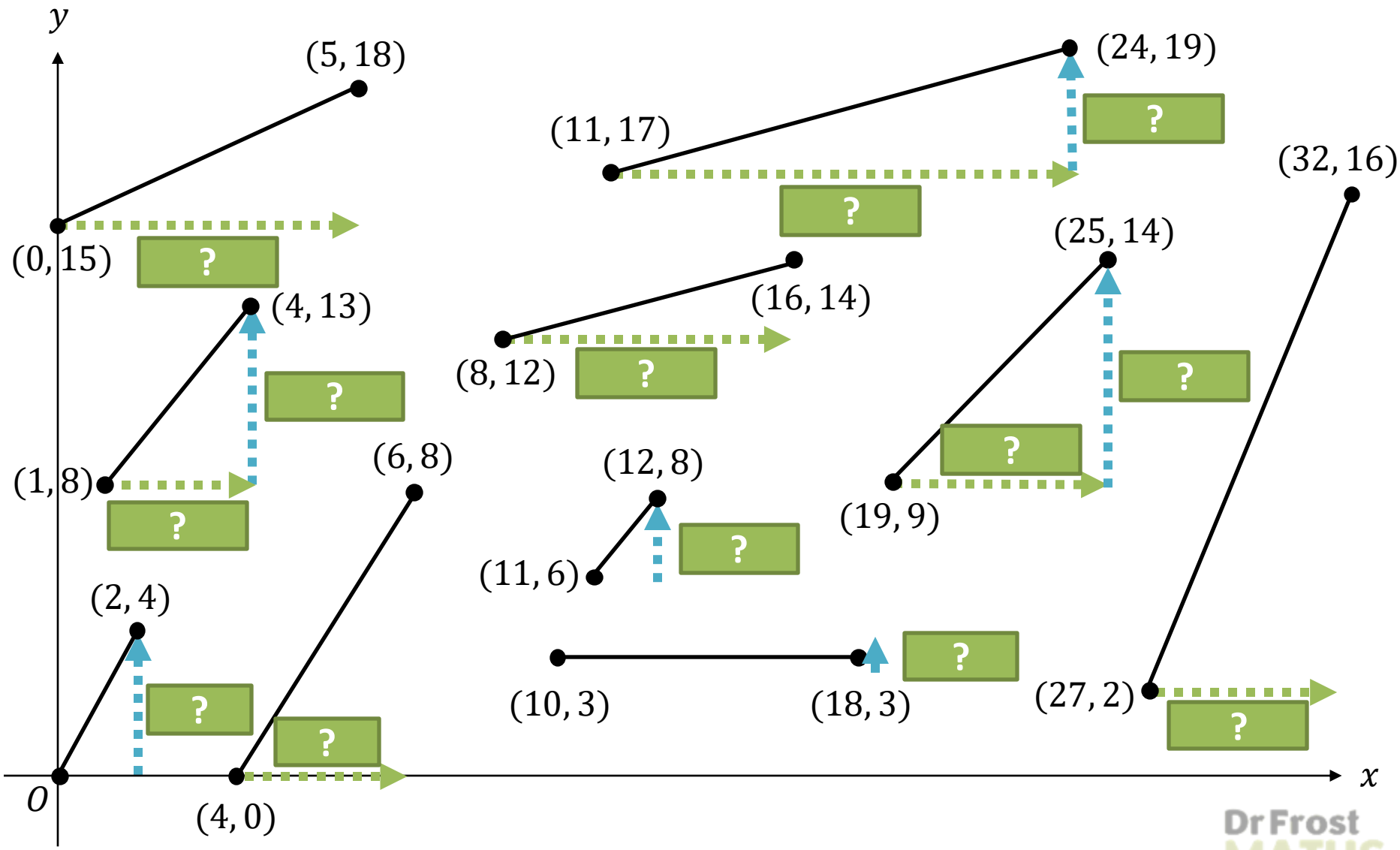
Remember to add the horizontal distance to the x-coordinate & the vertical distance to the y-coordinate

$$(2 + 2, 7 + 4)$$

$$\mathbf{(4, 11)}$$

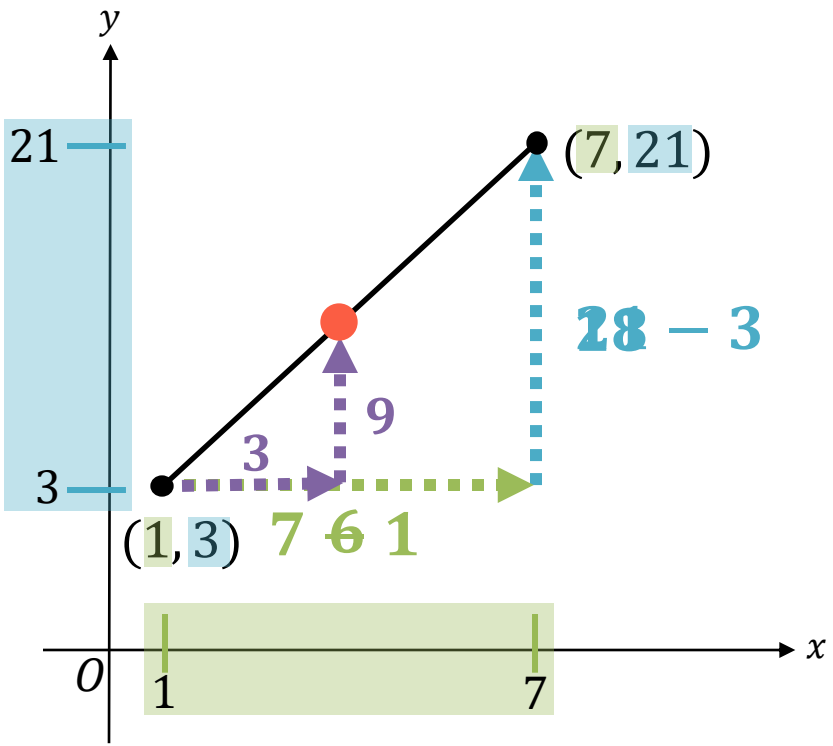
Quickfire Questions

Find the difference calculations between the x- & y-coordinates



Example

Find the midpoint of the line segment.

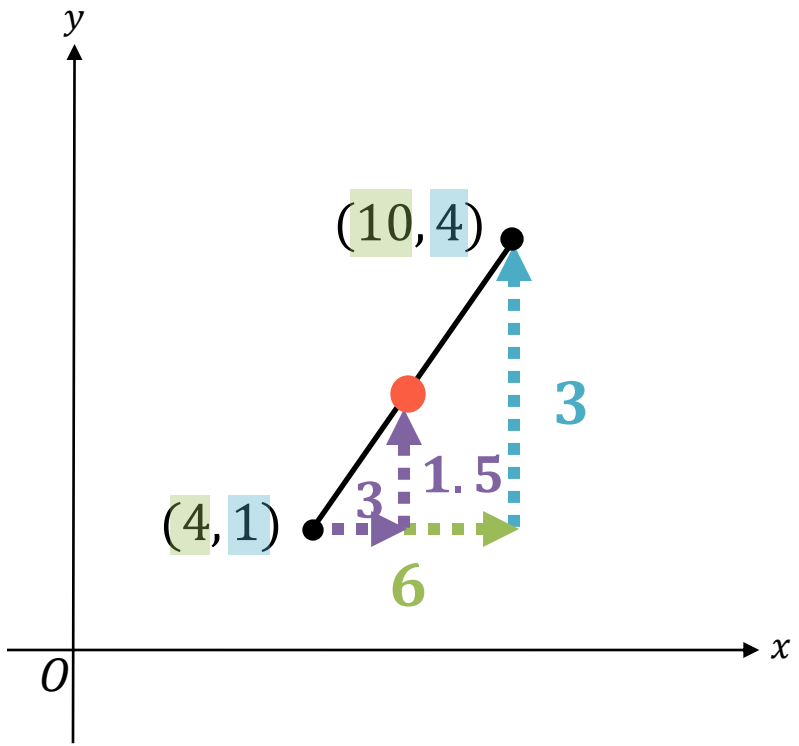


$$(1 + 3, 3 + 9)$$

$$\mathbf{(4, 12)}$$

Test Your Understanding

Find the midpoint of the line segment.



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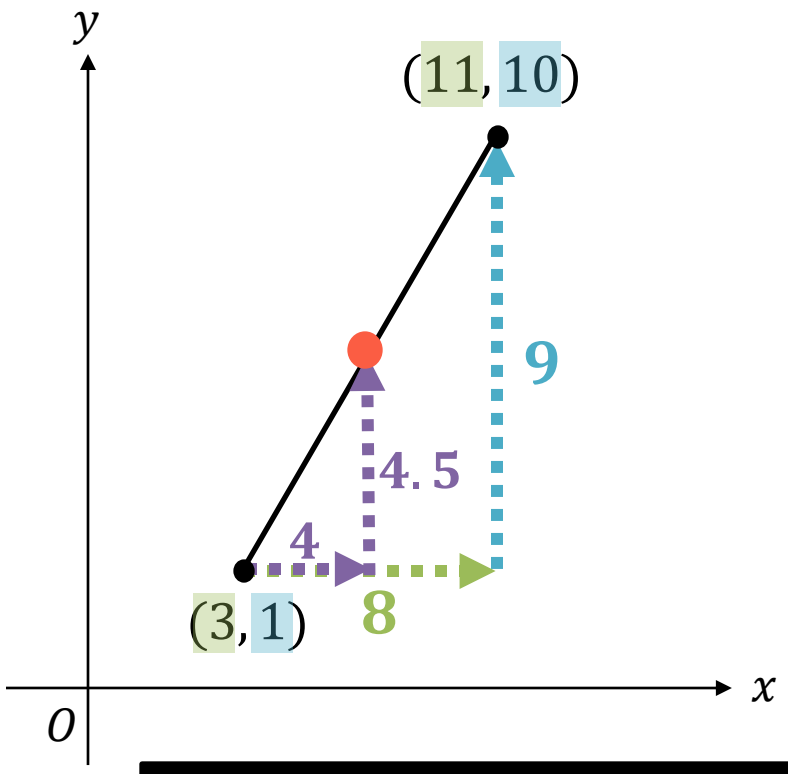
Spot the Mistake

Maria has completed her homework but has made a mistake.

Find the midpoint of the coordinates $(3, 1)$ & $(11, 10)$.



$$(3 + 4.5, 1 + 4)$$
$$(\mathbf{7.5}, \mathbf{5})$$

What is the mistake that Maria has made?

Maria has added the horizontal distance to the y-coordinate rather than the x-coordinate & vice versa

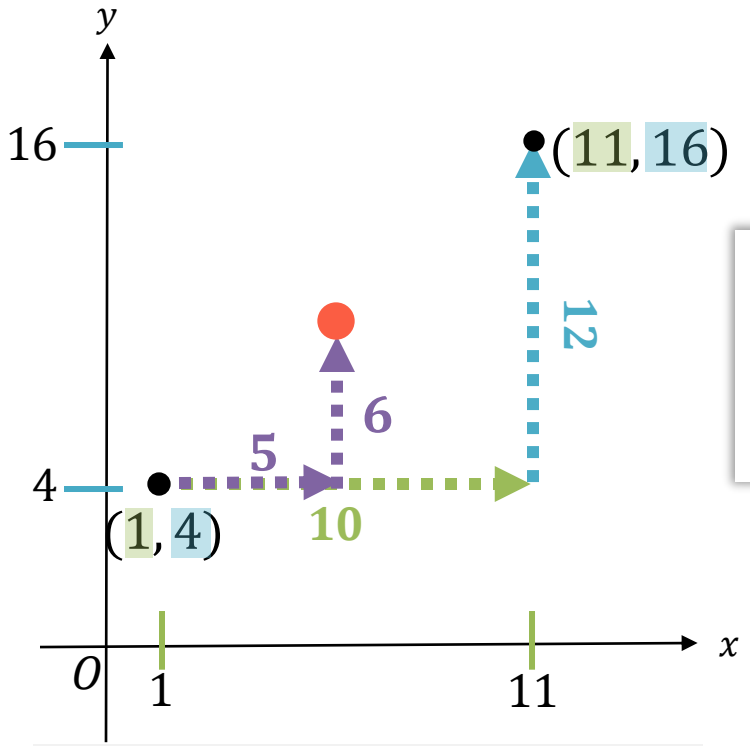
The correct answer is...

$$(3 + 4, 1 + 4.5)$$
$$(\mathbf{7}, \mathbf{5.5})$$

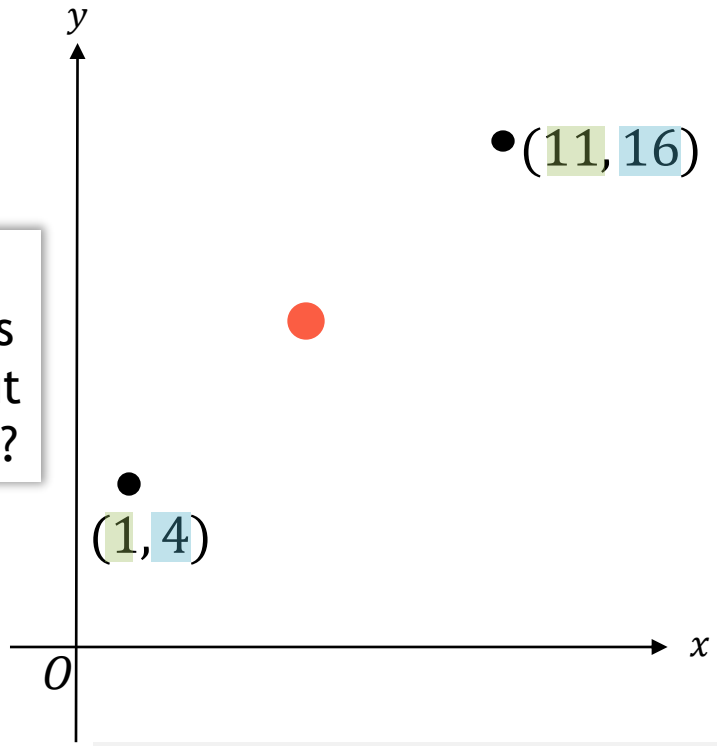
Remember to add the horizontal distance to the x-coordinate & the vertical distance to the y-coordinate

What's the Same? What's Different?

Hannah and Logan have both found the midpoint of two coordinates, but they have both approached it in different ways.



What's the same & what's different about their methods?



$11 - 1 = 10$ $16 - 4 = 12$

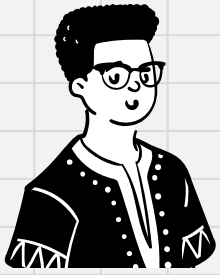
Hannah

$10 \div 2 = 5$

$12 \div 2 = 6$

$(1 + 5, 4 + 6)$

$(6, 10)$



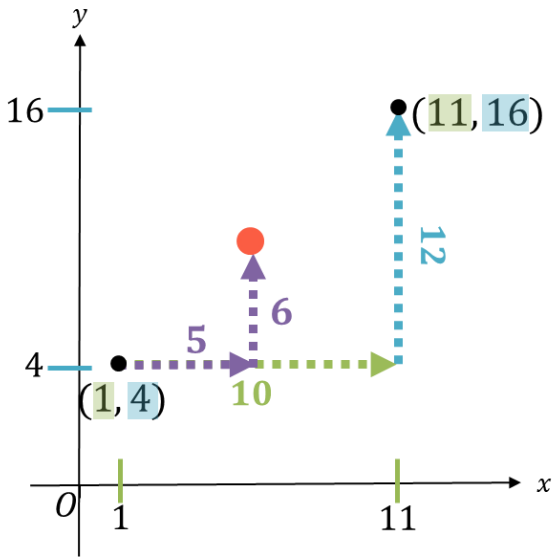
Logan

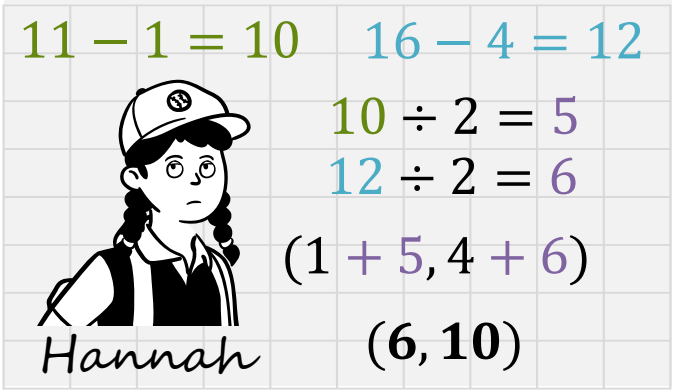
$\left(\frac{1+11}{2}, \frac{4+16}{2}\right)$

$\left(\frac{12}{2}, \frac{20}{2}\right)$

$(6, 10)$

What's the Same? What's Different?



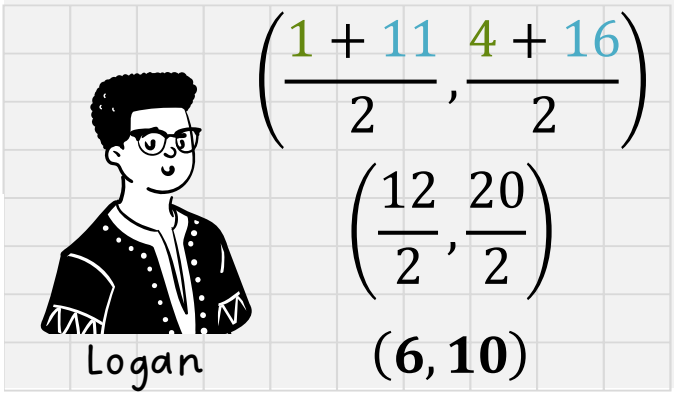


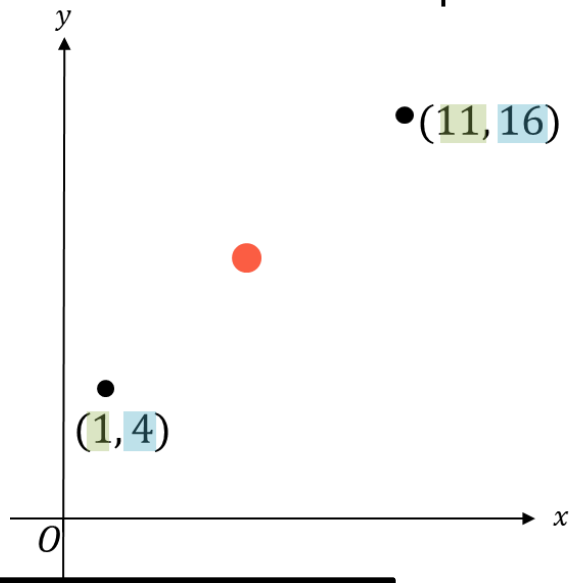
Hannah has found the distances between the x- & y-coordinates by finding the differences.

They have then halved these distances and added them to the initial endpoint because they know that the midpoint is **exactly halfway** between the two endpoints.

Logan has summed both x- and y-coordinates respectively, then divided by 2.

He has realised that the **average** of the x- & y-coordinates will be **exactly halfway** between the endpoints.



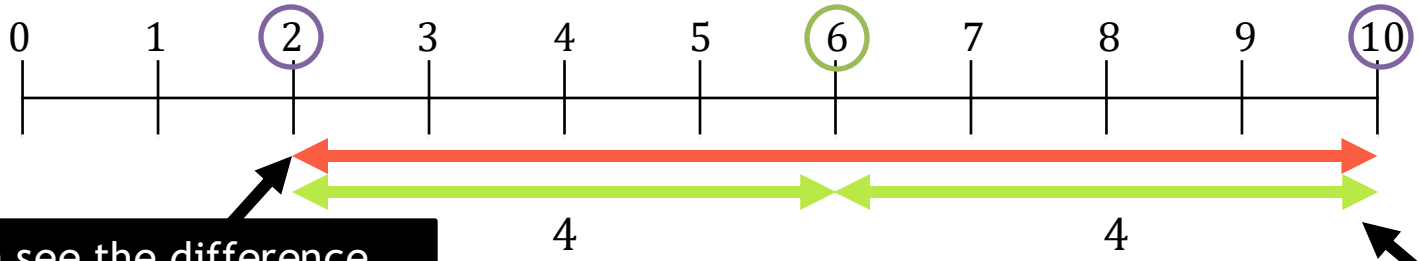


Finding the midpoint of two coordinates is the same as finding the midpoint of two numbers, in **two dimensions** rather than one!

RECAP: Finding the Midpoint of Two Numbers

Find the midpoint of 2 and 10.

This is the number exactly halfway between 2 and 10 - It is the middle number



We can see the difference between the two numbers is 8

Therefore, the midpoint has a difference of 4 from each endpoint

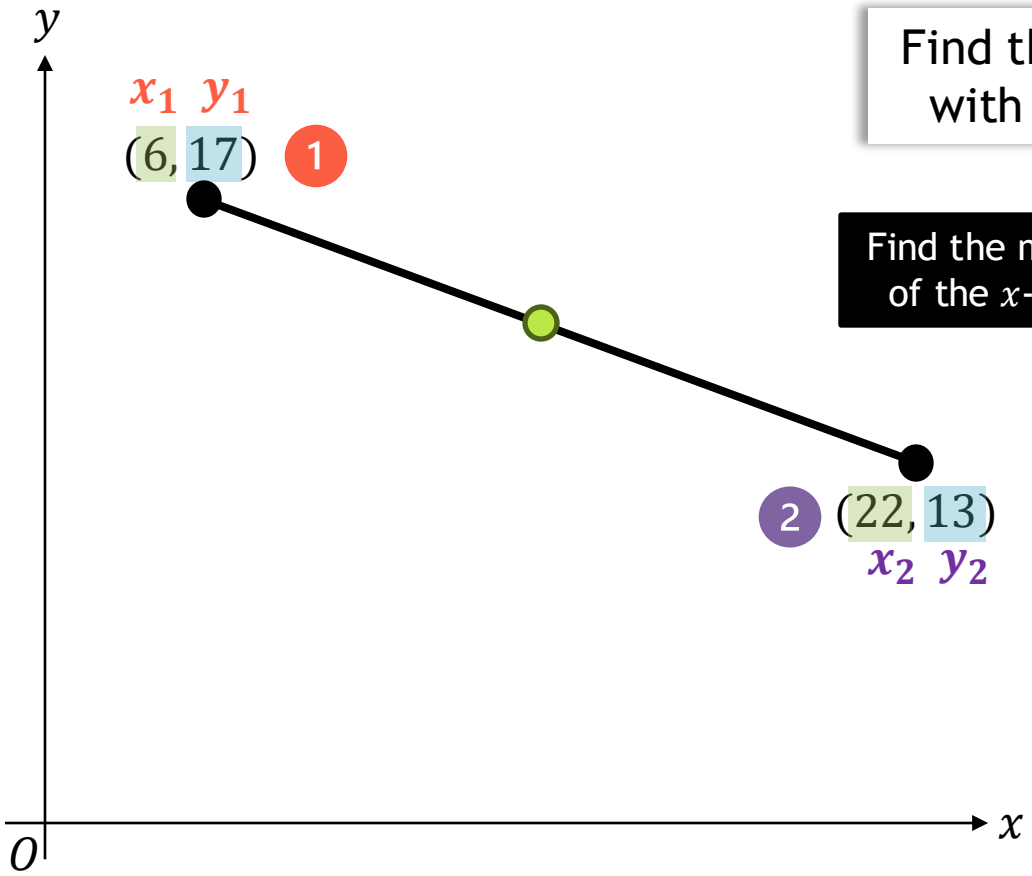
As the midpoint can also be thought of as the average of the two points, it can be calculated by the finding of the two values, and dividing by 2

$$\frac{2+10}{2} = \frac{12}{2} = 6$$

This is finding the midpoint in one dimension

The Big Idea: The Midpoint of Any Two Points

If we extend this into two dimensions (i.e. on and x- & y-axis), we can apply the same method to find the midpoint in each dimension.



Find the midpoint of the line segment with endpoints $(6, 17)$ and $(22, 13)$.

Find the mean average of the x-coordinates

Find the mean average of the y-coordinates

$$\left(\frac{6+22}{2}, \frac{17+13}{2}\right)$$

$$\left(\frac{28}{2}, \frac{30}{2}\right)$$

$$(\mathbf{14}, \mathbf{15})$$

We can generalise this to give…

The midpoint of two coordinates (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Examples

1 Find the midpoint of the coordinates $(1, 8)$ and $(3, 2)$.

$$\left(\frac{1+3}{2}, \frac{8+2}{2}\right)$$

$$\left(\frac{4}{2}, \frac{10}{2}\right)$$

$$(\mathbf{2}, \mathbf{5})$$

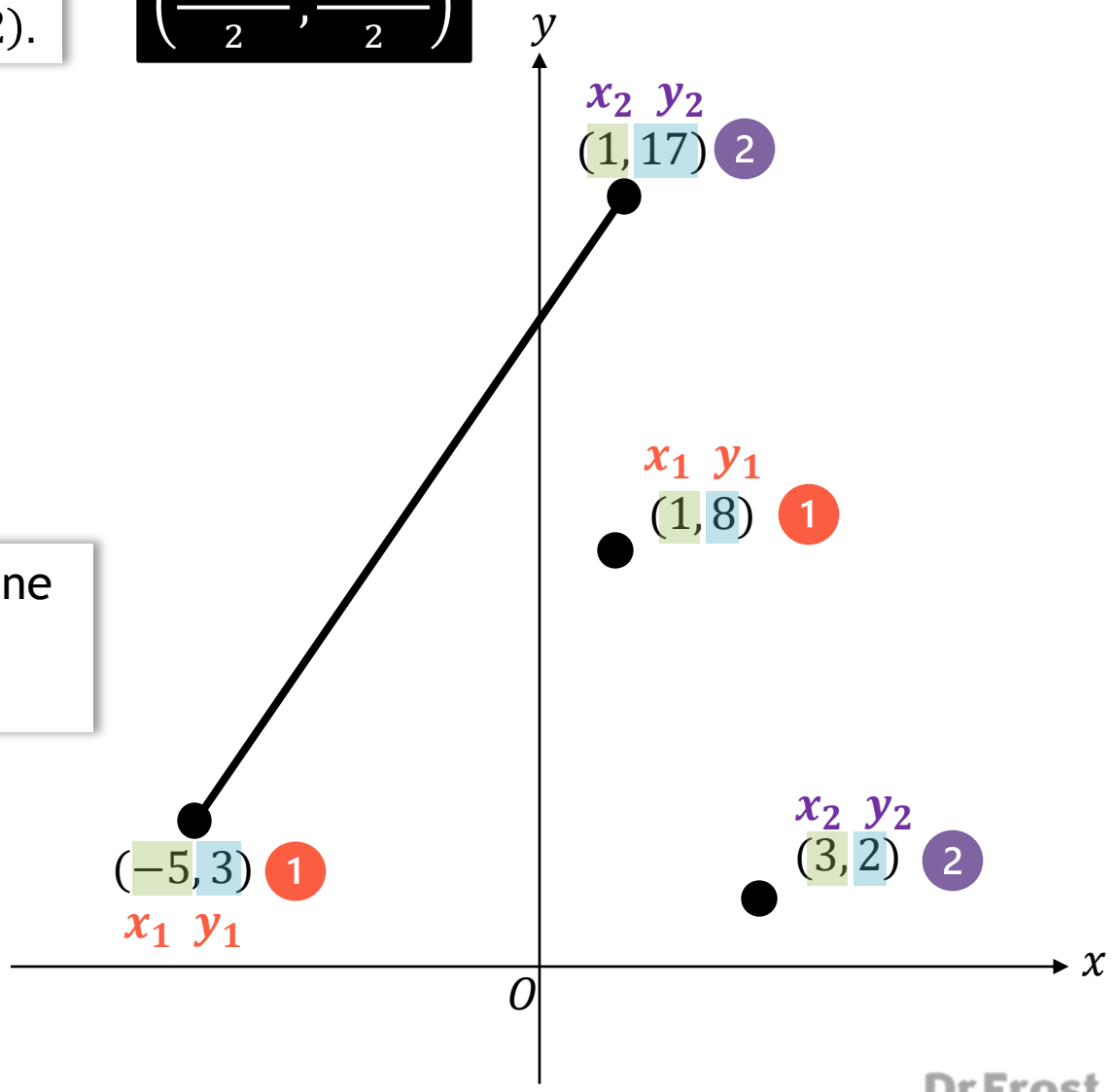
2 Find the midpoint of the line segment with endpoints $(-5, 3)$ and $(1, 17)$.

$$\left(\frac{-5+1}{2}, \frac{3+17}{2}\right)$$

$$\left(\frac{-4}{2}, \frac{20}{2}\right)$$

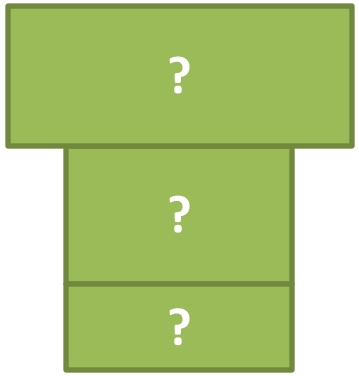
$$(\mathbf{-2}, \mathbf{10})$$

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

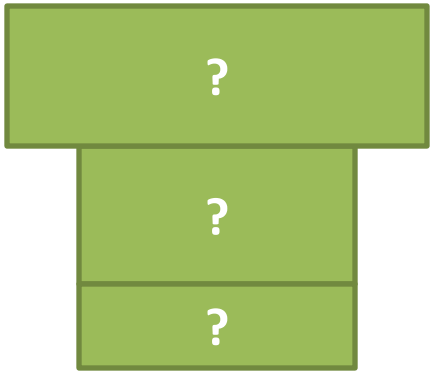


Test Your Understanding

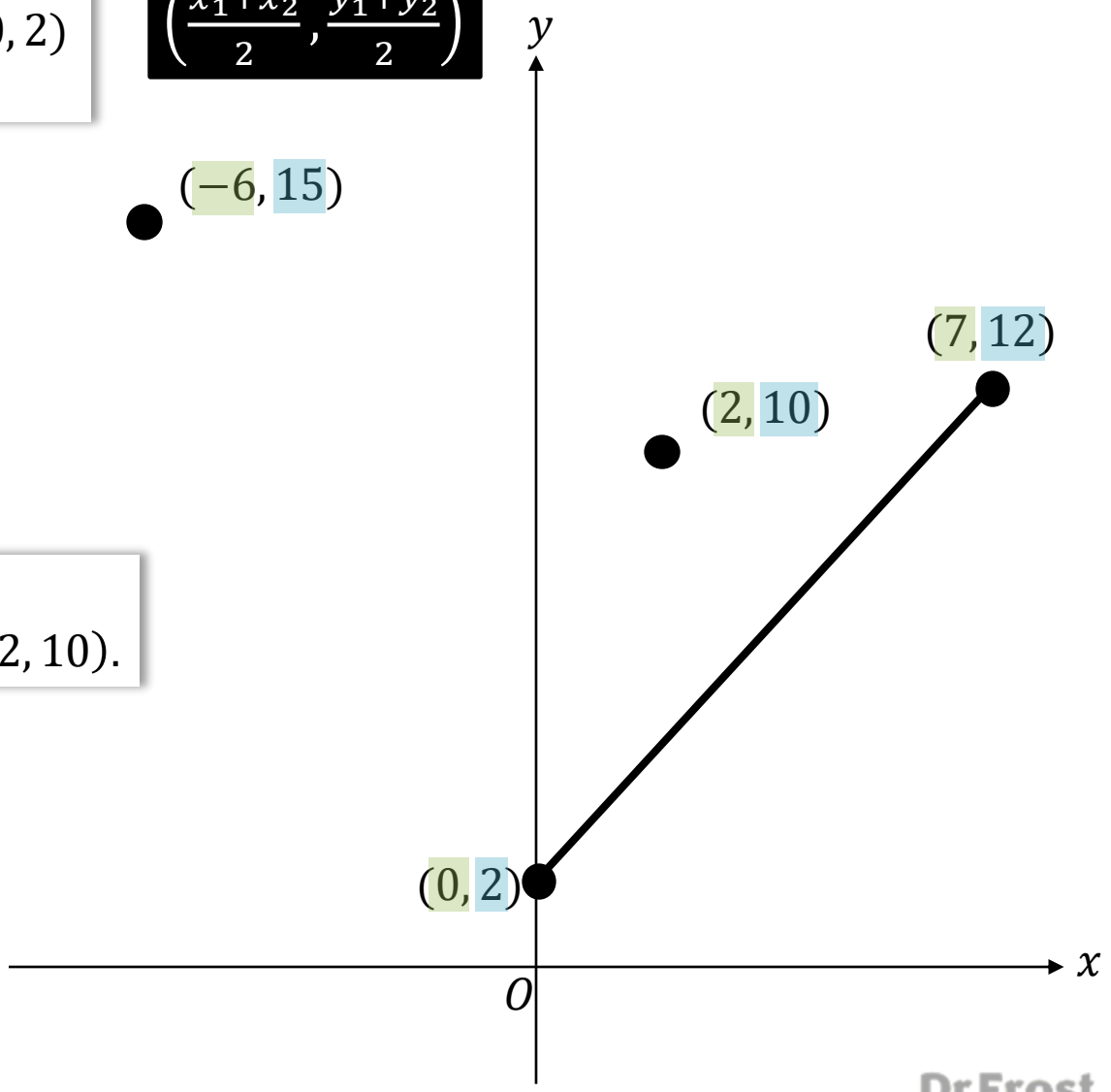
1 Find the midpoint of the line segment with endpoints $(0, 2)$ and $(7, 12)$.



2 Find the midpoint of the coordinates $(-6, 15)$ and $(2, 10)$.

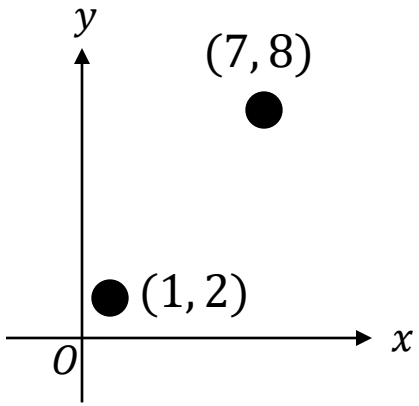


$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$



Quickfire Questions

What is the midpoint of...



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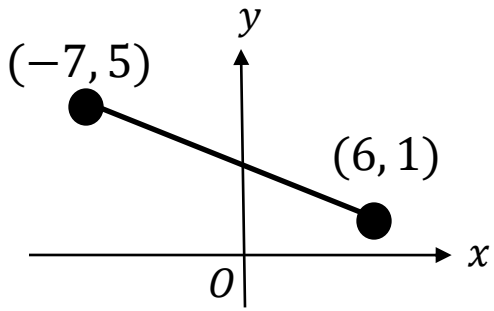
What is the midpoint of...



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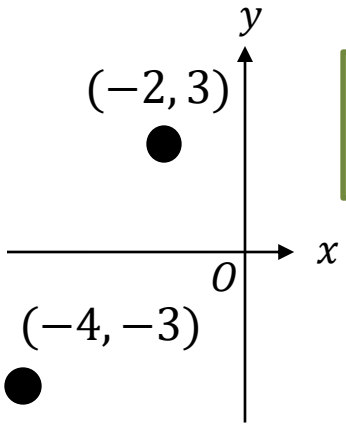
What is the midpoint of...



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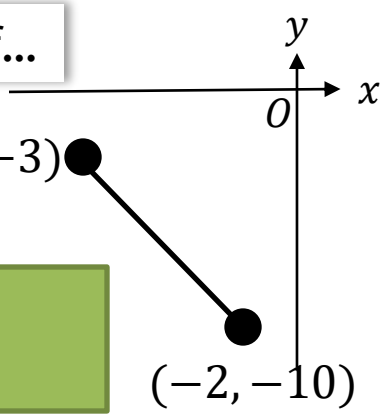
What is the midpoint of...



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Dr Frost MATHS

Problem Solving

P is the midpoint of AB.
Q is the midpoint of AC.
M is the midpoint of PQ.
Find the coordinates of the point M.

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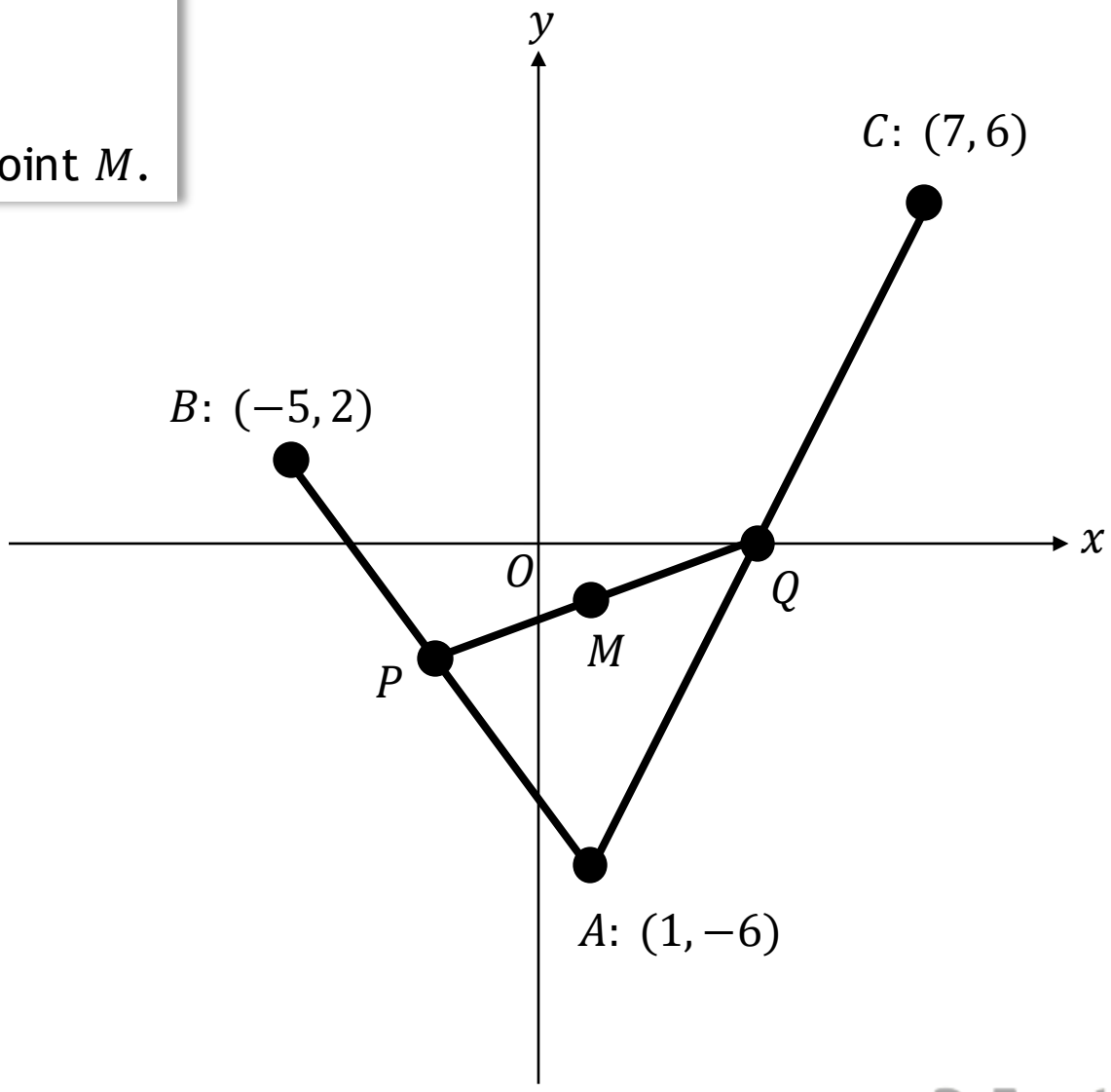
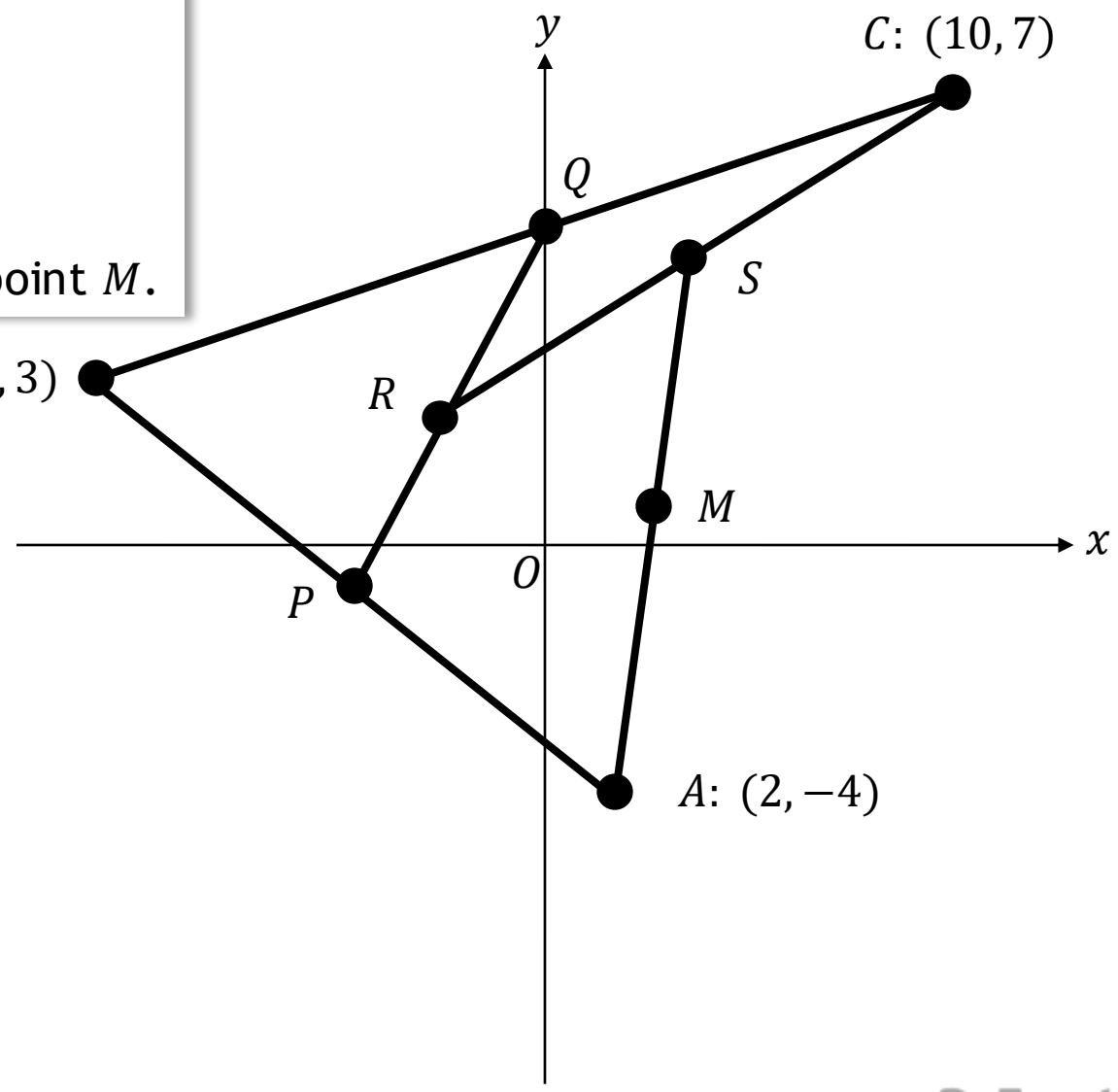


Problem Solving

P is the midpoint of AB.
Q is the midpoint of BC.
R is the midpoint of PQ.
S is the midpoint of RC.
M is the midpoint of AS.
Find the coordinates of the point M.



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Worded Questions

[Edexcel IGCSE Jan 2020 1F Q14a, Edexcel IGCSE Jan 2020 1H Q1a]

The point A has coordinates $(5, -4)$.
x_1 y_1

The point B has coordinates $(13, 1)$.
x_2 y_2

Work out the coordinates of the midpoint of AB.

$$\left(\frac{5+13}{2}, \frac{-4+1}{2}\right)$$

$$\left(\frac{18}{2}, \frac{-3}{2}\right)$$

$$\mathbf{(9, -1.5)}$$

Often, questions that we can come across regarding midpoint may not have a diagram

But we can still use the same technique of finding the average of the x- and y-coordinates:

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Test Your Understanding

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[CCEA GCSE Jan 2015 T2 Foundation Q20]

P is the point $(2, 3)$ and Q is the point $(-4, -1)$.
Work out the coordinates of the midpoint of the line PQ.

x_1 y_1 x_2 y_2

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Who is Correct?

Try the following question...

Professor Cheng

Find the midpoint of the line segment that goes through the coordinates $(-4, 3)$ & $(10, 7)$.

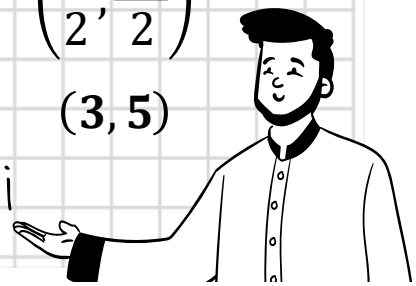
Who is correct and why?

$$\left(\frac{-4+10}{2}, \frac{3+7}{2}\right)$$

$$\left(\frac{6}{2}, \frac{10}{2}\right)$$

$$(\mathbf{3}, \mathbf{5})$$

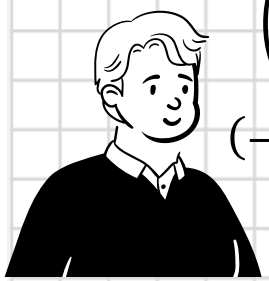
Abdi



$$\left(\frac{-4+3}{2}, \frac{10+7}{2}\right)$$

$$\left(\frac{-1}{2}, \frac{17}{2}\right)$$

$$(\mathbf{-0.5}, \mathbf{8.5})$$



VIKTOR

Abdi, H... or have all a... that Pr... d.

$$\left(\frac{10+-4}{2}, \frac{7+3}{2}\right)$$

$$\left(\frac{6}{2}, \frac{10}{2}\right)$$

$$(\mathbf{3}, \mathbf{5})$$

Basma

$$\left(\frac{-4-10}{2}, \frac{3-7}{2}\right)$$

$$\left(\frac{-14}{2}, \frac{-4}{2}\right)$$

$$(\mathbf{-7}, \mathbf{2})$$

Hannah

Who is Correct?

Abdi has labelled his coordinates as...

Find the midpoint of the line segment that goes through the coordinates $(-4, 3)$ & $(10, 7)$.

x_1 y_1 x_2 y_2

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Basma has her coordinates labelled as...

Find the midpoint of the line segment that goes through the coordinates $(-4, 3)$ & $(10, 7)$.

x_2 y_2 x_1 y_1

Abdi and Basma are both correct!

But what have they done differently?

How are they both correct?

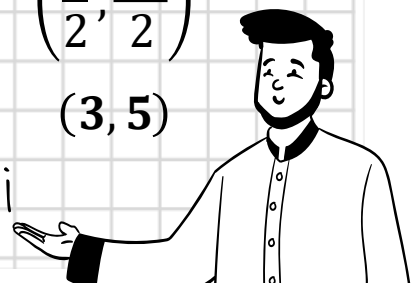
As addition is commutative, the x- or y-coordinates can be summed in any order

$$\left(\frac{-4+10}{2}, \frac{3+7}{2}\right)$$

$$\left(\frac{6}{2}, \frac{10}{2}\right)$$

$$(3, 5)$$

Abdi



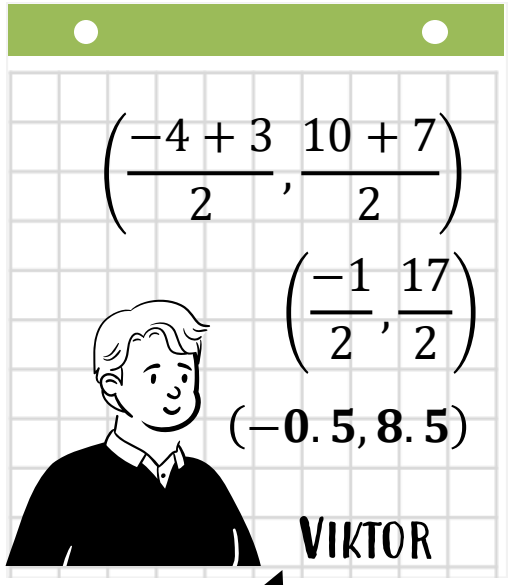
$$\left(\frac{10+4}{2}, \frac{7+3}{2}\right)$$

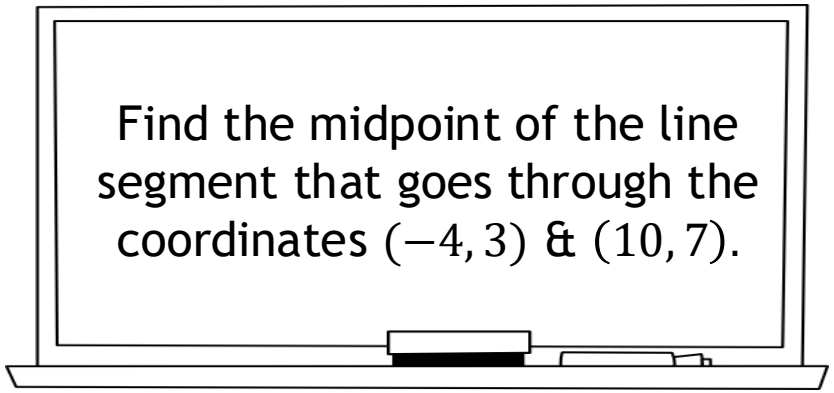
$$\left(\frac{6}{2}, \frac{10}{2}\right)$$

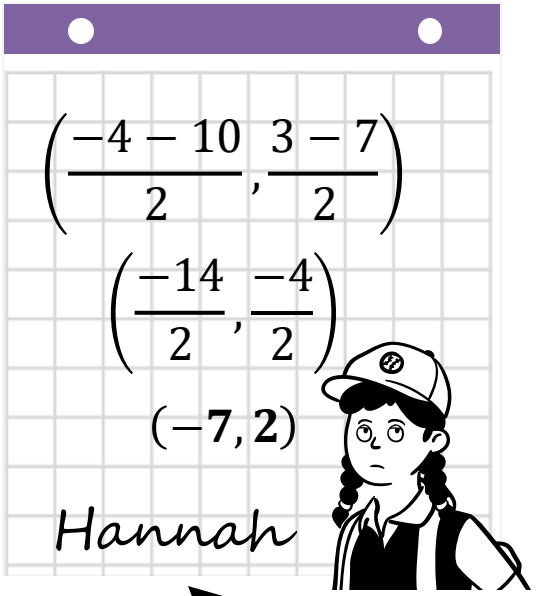
$$(3, 5)$$

Basma

Who is Correct?







Viktor and Hannah are both incorrect.

What have they done incorrectly?

Viktor has summed the x- and y-coordinates of each point together, rather than summing together each x-coordinate, then each y-coordinate

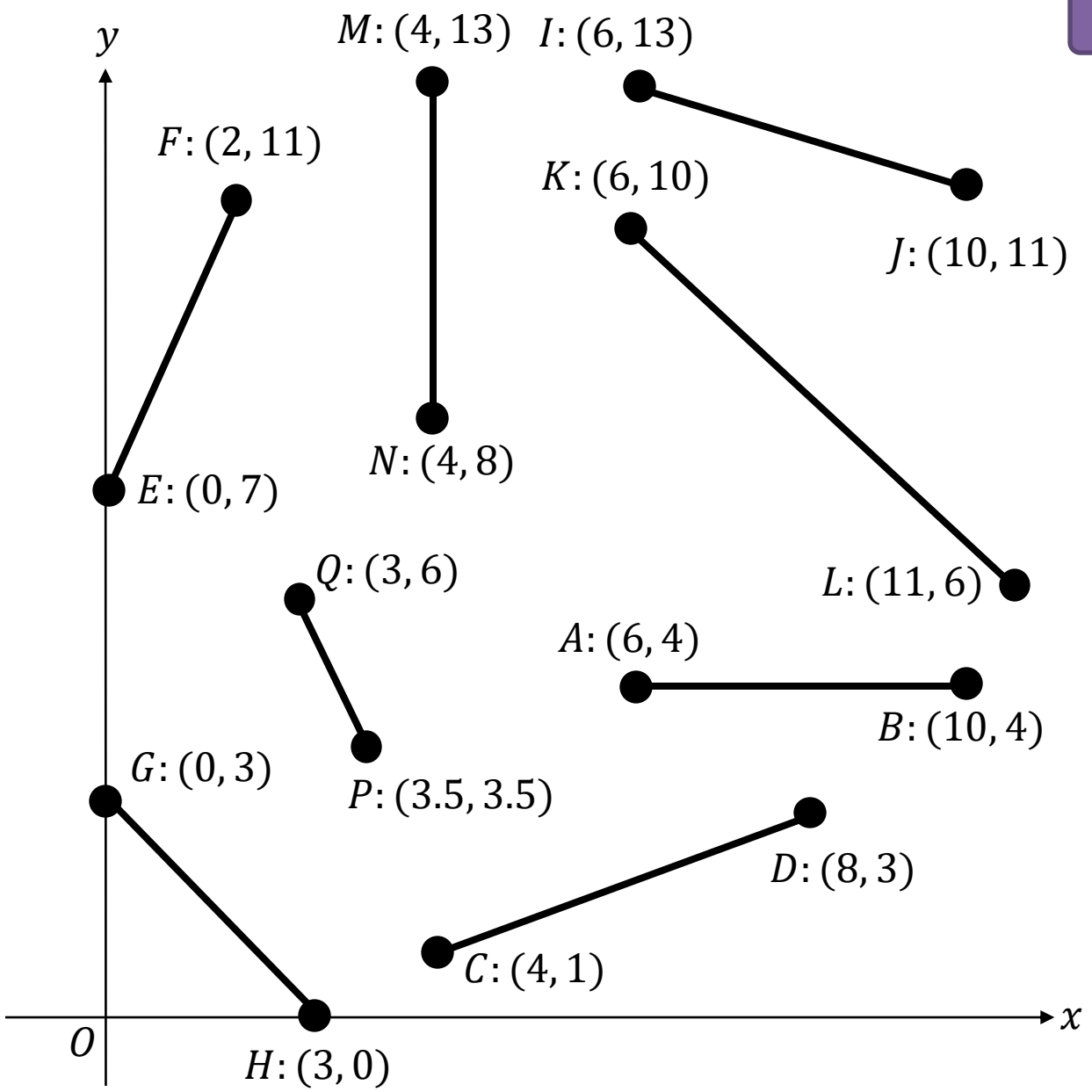
Hannah has found the difference in the x- and y-coordinates before halving them, rather than summing. They are incorrectly finding the average of the x- and y-coordinates

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Exercise

(Available as a separate worksheet)

Show all solutions



1 Find the midpoints of the line segments on the coordinate grid:

a M_{AB}: ?

b M_{CD}: ?

c M_{EF}: ?

d M_{GH}: ?

e M_{IJ}: ?

f M_{KL}: ?

g M_{MN}: ?

h M_{PQ}: ?

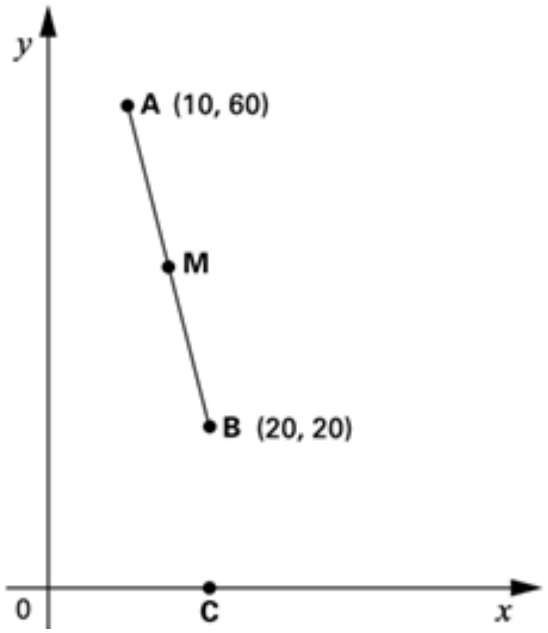
Exercise

(Available as a separate worksheet)

Show all solutions

2 *[KS2 SATs 2001 Paper 1 Q20a,b]*

$\boldsymbol{A}$ is the point $(10, 60)$
$\boldsymbol{B}$ is the point $(20, 20)$
$\boldsymbol{M}$ is the midpoint of line AB.



a Write the coordinates of $\boldsymbol{M}$.

?

b $\boldsymbol{C}$ is on the x-axis, directly **below B**.
Write the coordinates of $\boldsymbol{C}$.

?

3 *[Edexcel IGCSE Jan2017(R)-3H Q3]*

A is the point with coordinates $(4, 11)$
B is the point with coordinates $(8, 3)$
Work out the coordinates of the midpoint of AB.

?

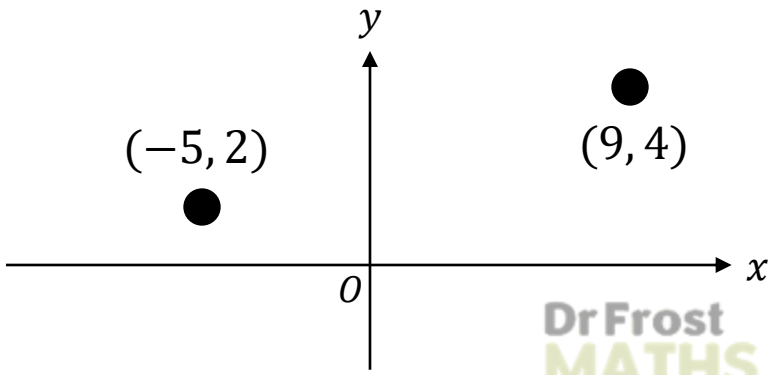
4 *[Edexcel IGCSE Jan2015(R)-3H Q6]*

A is the point with coordinates $(4, 1)$.
B is the point with coordinates $(1, 9)$.
Find the coordinates of the midpoint of AB.

?

5 Find the midpoint of the two coordinates

?



Exercise

(Available as a separate worksheet)

Show all solutions

6 Find the midpoints of the line segments on the coordinate grid:

a M_{AB}: ?

b M_{CD}: ?

c M_{EF}: ?

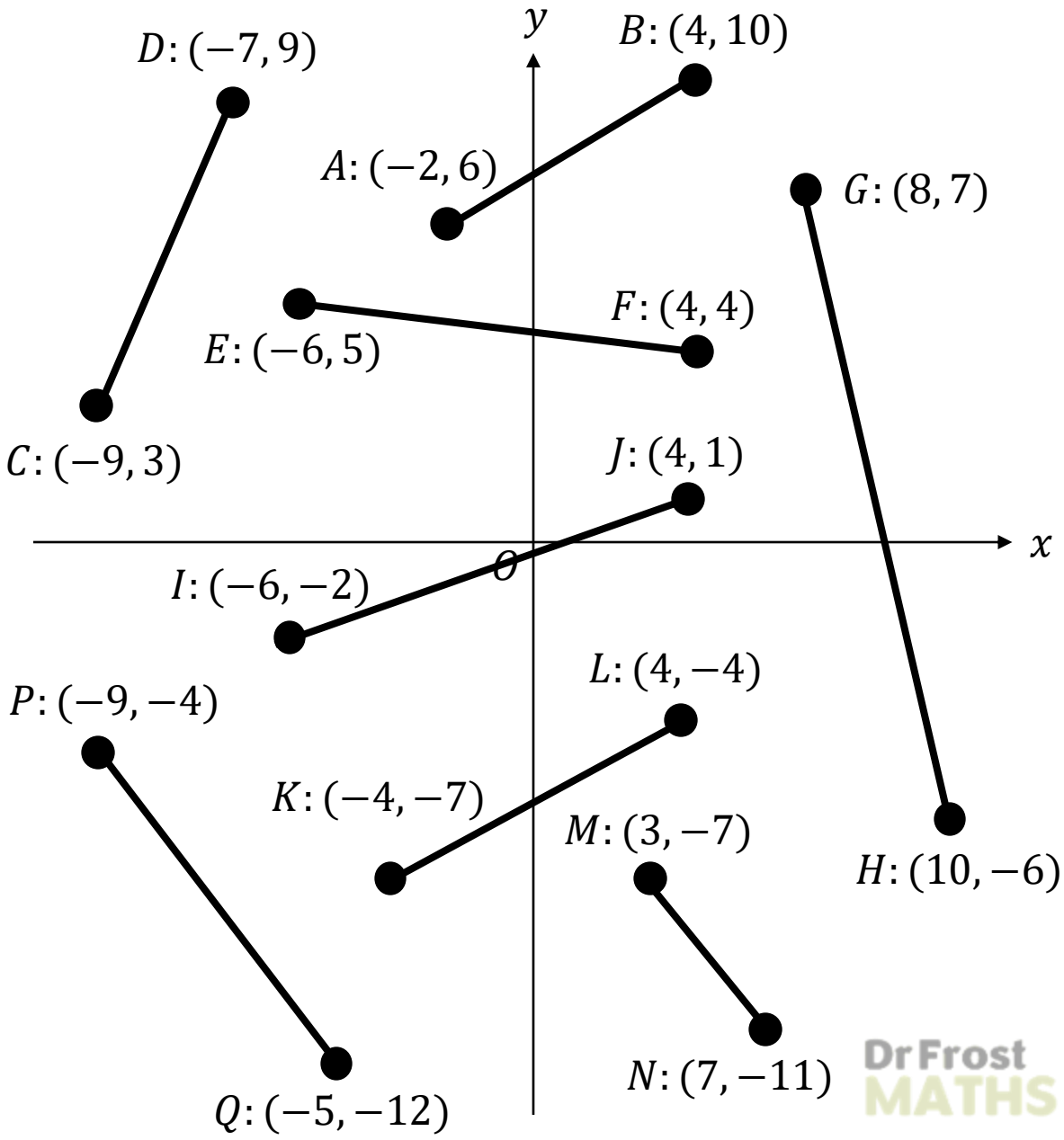
d M_{GH}: ?

e M_{IJ}: ?

f M_{KL}: ?

g M_{MN}: ?

h M_{PQ}: ?



Dr Frost MATHS

Exercise

(Available as a separate worksheet)

Show all solutions

7 *[Edexcel IGCSE(9-1) SAM 2F Q14]*

A has coordinates $(3, 6)$

B has coordinates $(-5, 8)$

Work out the coordinates of the midpoint of AB.

?

8 X has coordinates $(-8, -2)$

Y has coordinates $(-3, 5)$

Work out the coordinates of the midpoint of XY.

?

9 *[WJEC AS Level Summer 2019 Unit 1 Q9a Edited]*

The points $A(-2, 4)$ and $B(6, 10)$ are such that AB is the diameter of a circle.

Find the centre of the circle.

?

X is the midpoint of AB.

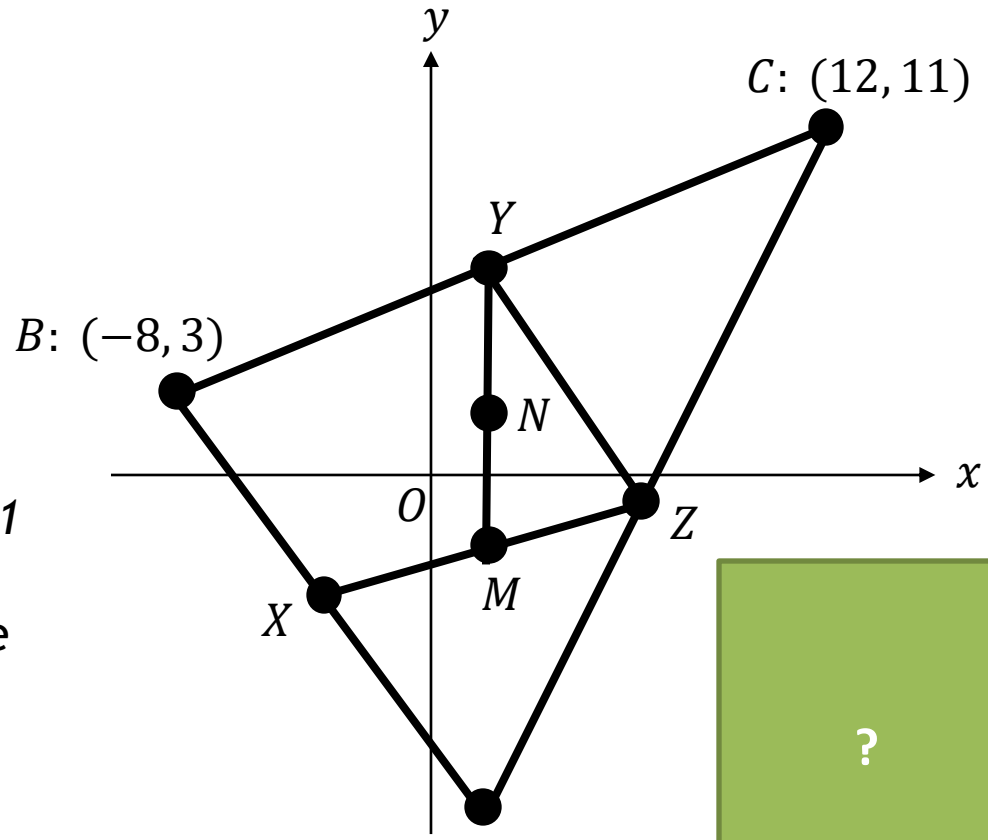
Y is the midpoint of BC.

Z is the midpoint of AC.

M is the midpoint of XZ.

N is the midpoint of YM.

Find the coordinates of the point N.



?

CHALLENGE QUESTIONS

1. Parameterised Points (Algebraic)

Points $A(2t,\ 3t+1)$ and $B(4-t,\ 5)$ have midpoint $M(3,\ 4)$.
Find t.

---

2. Reverse Engineering Coordinates

The midpoint of $A(1, 7)$ and $B(x, y)$ is $(5, 3)$.
Find x and y.

---

3. Collinearity with Midpoint

Points $A(2, -1)$, $B(6, 7)$, and M is the midpoint of AB.
Show that M lies on the line $y = 2x - 1$.

1. Parameterised Points

Midpoint:

$$\left(\frac{2t + (4 - t)}{2}, \frac{3t + 1 + 5}{2}\right) = (3, 4)$$

Solve:

- $\frac{t+4}{2} = 3 \Rightarrow t = 2$
- Check y: $\frac{3t+6}{2} = 4 \Rightarrow t = 2$

Answer: $t = 2$

---

2. Reverse Engineering Coordinates

$$\frac{1 + x}{2} = 5 \Rightarrow x = 9, \quad \frac{7 + y}{2} = 3 \Rightarrow y = -1$$

Answer: $(x, y) = (9, -1)$

---

3. Collinearity

Midpoint:

$$M = \left(\frac{2 + 6}{2}, \frac{-1 + 7}{2}\right) = (4, 3)$$

Check line $y = 2x - 1$:
$2(4) - 1 = 7 \neq 3$

Answer: False statement — midpoint does *not* lie on the line.

TRUE or FALSE

The midpoint lies exactly halfway between two points on a straight line.

TRUE

TRUE or FALSE

If the midpoint of a line segment is (3,5) and one endpoint is (1,2), the other endpoint must be (5,8).

TRUE